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YIELD-POINT LOAD DETERMINATION BY NONLINEAR PROGRAMMING

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Abstract

The determination of the yield-point load of a perfectly plastic structure can be formulated as a nonlinear programming problem by means of the theorems of limit analysis. This formulation is discussed in general terms and then applied to the problem of a curved beam. Recent results in the theory of nonlinear programming are called upon to solve typical problems for straight and curved beams. The theory of limit analysis enables intermediate answers to be given a physical interpretation in terms of upper and lower bounds on the yield-point load. The paper closes with some indication of how the method may be generalized to more complex problems of plastic yield-point load determination.

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1. Introduction

The constitutive behavior of an ideal elastic/perfectly-plastic material is defined in terms of a yield function $f(\sigma_{ij})$. The stress tensor σ_{ij} must be such that

$$f(\sigma_{ij}) \leq 0 \tag{1}$$

and the strain rates $\dot{\epsilon}_{ij}$ are then given by

$$\dot{\epsilon}_{ij} = C_{ijk\ell}\dot{\sigma}_{k\ell} + \dot{\lambda}\,\partial f/\partial\sigma_{ij} \tag{2}$$

where

$$\dot{\lambda} \geq 0 \quad . \tag{3}$$

and

$$\dot{\lambda} = 0 \quad \text{if} \quad f < 0 \quad \text{or} \quad \dot{f} < 0 \tag{4}$$

Here the $C_{ijk\ell}$ are the elastic constants of the generalized Hooke's law, $\dot{\lambda}$ is an unknown scalar function, and a dot over a symbol indicates differentiation with respect to time.

The general elastic-plastic problem is concerned with a structure or body made of an ideal elastic/perfectly-plastic material subjected to a given set of surface tractions* P_i which are prescribed at all points of the surface except where the corresponding velocity is prescribed to be zero. A solution consists in finding a stress tensor σ_{ij}, and a velocity vector v_i such that

(a) the stresses are in internal equilibrium

$$\sigma_{ji,j} = 0 \tag{5}$$

(b) for a given constant S, the stresses are in equilibrium with loads SP_i on the boundary

$$\sigma_{ji} n_j = SP_i \tag{6}$$

(c) the yield inequality (1) is valid;

(d) the velocity satisfies any boundary constraints, and the strain rate field derived from it satisfies (2),(3) and (4).

This problem may be viewed as a boundary value problem, but such an approach involves several difficulties among which are the following.

A different set of differential equations must be solved in the "elastic region" ($f < 0$ or $\dot{f} < 0$) and "plastic region" ($f = \dot{f} = 0$). Further, the plastic region equations are nonlinear.

The location of the elastic-plastic interface between the two regions is an unknown of the problem.

Continuity requirements across the interface and within the plastic region are not entirely straightforward.

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*For simplicity of exposition, body forces are neglected; this restriction is not vital to the material that follows.

If S is less than a certain critical value S_o, known as the safety factor or yield-point load, (and which is not known a priori) a unique solution to the problem exists. However, for $S > S_o$, no solution exists. For $S = S_o$ a solution exists for which the magnitude of the velocity vector is not unique. Further, in many problems there will be further lacks of uniqueness associated with the case $S = S_o$. Therefore, solution of the boundary value problem for a given S depends upon the relation of S to the unknown S_o for such vital properties as existence and uniqueness.

Within the boundary-value-problem approach, S_o can be determined by first solving for an arbitrarily small S for which the entire solution will be elastic. Since the elastic solution is linear in S, the maximum $S = S_1$ for which the fully elastic solution holds is easily determined by observing the largest S for which this solution satisfies (1) everywhere. One then solves a sequence of problems with $S = S_1 + K\,\Delta S$, $K = 1,2,\ldots$ for as long as a solution exists, deducing that

$$S_1 + (K_1 - 1)\Delta S \leq S_o \leq S_1 + K_1\Delta S \qquad (7)$$

where K_1 is the smallest K for which no solution exists. Although easily stated, the above problem is obviously far from trivial in any but the very simplest of cases.

Although only a very few simple problems have been completely solved, it is instructive to examine the behavior of a typical displacement d of the problem as a function of the load-magnitude S. Figure 1 shows the qualitive behavior that is always present. For $S < S_1$, d is a linear function of S. As S increases above S_1, d increases (generally non-linearly) at an increasing rate, tending to infinity as S tends to S_o. Since infinite displacements are rarely admissible, it appears that the value of S_o is of crucial importance in the analysis of the problem. From the mathematical viewpoint knowledge of its value is a necessary prerequisite to proper posing of the problem. From a practical viewpoint in many applications, determination of S_o may be the primary question of interest, and additional effort spent on finding σ_{ij} and v_i may be unwarranted.

To summarize, if one uses a boundary-value-problem approach to a practical problem in plasticity, one must solve a non-linear, free boundary, difficult to pose problem; if one is successful one ends up with the desired number S_o together with a stress and displacement distribution which may not be required.

The Theorems of Limit Analysis* provide an alternative approach to the determination of S_o. The Lower Bound Theorem, with which we will be chiefly concerned, states that if for any number S^- there exists a stress tensor σ_{ij} which satisfies requirements (a), (b) and (c) listed earlier, i.e., which is in internal and external equilibrium with S^-P_i and does not violate (1), then S^- is a lower bound on S_o:

$$S^- \leq S_o \tag{8}$$

*The theorems were independently discovered by Gvozdev [1], Drucker, Greenberg, and Prager [2,3,4], and Hill [5,6]. Textbook accounts may be found in [7,8,9].

Since a solution to the complete problem is known to exist for $S = S_o$, this theorem establishes the uniqueness of S_o and may be restated in slightly stronger form by saying that S_o is the maximum of the set of numbers for which (1), (5) and (6) possess a solution:

$$S_o = \max(S \mid \sigma_{ji,j} = 0,\ \sigma_{ji}n_j = SP_i,\ f(\sigma_{ij}) < 0) \tag{9}$$

The similarity of this formulation to a programming problem suggests that techniques of mathematical programming may be of value in the solution of plasticity problems. That this is indeed the case will be demonstrated in the remaining sections of this paper.

The Upper Bound Theorem of limit analysis will not be used directly. However, we will make use of a corollary to this theorem which states that if the given structure is replaced by a "replacement structure" which is "stronger" (i.e., one whose yield inequality (1) is nowhere more restrictive than that of the given structure) and of the same size, shape, and loading, then the yield-point load of the replacement structure will not be less than that of the given structure.

2. Beam Problem as a Nonlinear Programming Problem

In order to make the discussion definite, a particular relatively simple problem will be considered, rather than the general three dimensional one. Specifically, we will consider a straight or curved beam subjected to in-plane loading.

According to beam theory, the stress state of any cross section of the beam is adequately specified by giving three numbers corresponding to the direct (axial) force N, the shear (transverse) force Q, and the bending moment M transmitted across the given section. Each of these quantities is a function of the arc length along the centroid of the beam. Further, it is assumed that the shear force does not noticeably affect the plastic behavior of the beam. Therefore analogous to (1) we have the requirement

$$F(N, M) \leq 0 \tag{10}$$

If, in particular, we specify a rectangular beam and choose suitable dimensionless variables* n, q, and m, then (10) becomes [10]

$$f(n,m) \equiv n^2 + |m| - 1 \leq 0 \tag{11}$$

Further discussion will be based on (11) although it will be evident that any more complex function would be as easily handled.

If the beam is subjected to normal and tangential loads P_n and P_s, respectively, then equilibrium in the axial and transverse directions and

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*Precise definitions of dimensionless quantities may be found the the Appendix.

moment equilibrium lead to

$$n' = -\kappa q - p_s$$

$$q' = \kappa n - p_n$$

$$m' = \eta q \tag{12}$$

in place of (5). Here primes indicate differentiation with respect to the dimensionless arc length $s, \kappa = \kappa(s)$ is the dimensionless curvature of the beam, p_n and p_s are dimensionless loads, and $\eta = \eta(s)$ is a known property of the cross-section dimensions (η = const. for a beam of uniform section).

For further definiteness we assume that the beam is fully constrained at either end. Then, since the reactions at the ends are not prescribed, there is no requirement analogous to (6).

The lower bound approach to the problem of determining the yield-point load may be formulated as follows: to determine the largest value α_o of α_1 for which there exists a solution to

$$n' + \kappa q + \alpha_1 p_s = q' - \kappa n + \alpha_1 p_n = m' - \eta q = 0 \tag{13}$$

which satisfies (11), α_o being the dimensionless equivalent of the yield-point load.

A programming problem is generally concerned with matrices of finite size, rather that functions which correspond to infinite matrices. The present problem may be conveniently reduced to finite size by taking advantage of the linearity of Eqs. (13). To this end, let $n_1(s)$, $q_1(s)$, $m_1(s)$ be the solution of Eqs. (12) (i.e., Eqs. (13) with $\alpha_1 = 1$) subjected to

$$n_1(0) = q_1(0) = m_1(0) = 0 \tag{14}$$

and let $n_j(s)$, $q_j(s)$, $m_j(s)$, $j = 2,3,4$ be the solution of the homogeneous counterparts of (13) subject to

$$n_2(0) = q_3(0) = m_4(0) = 1 \tag{15}$$

$$n_3(0) = n_4(0) = q_2(0) = q_4(0) = m_2(0) = m_3(0) = 0 \tag{16}$$

Then obviously

$$n(s) = \sum_{j=1}^{4} \alpha_j n_j(s) \qquad m(s) \sum_{j=1}^{4} \alpha_j m_j(s) \tag{17}$$

for any solution of (13) whatsoever. Therefore, regarding n_j and m_j as known functions of s which may be determined once and for all, we may formulate the problem as:

maximize α_1 subject to

$$\left(\sum \alpha_j n_j\right)^2 + \left| \sum \alpha_j m_j \right| - 1 \leq 0 \quad , \; 0 \leq s \leq \ell \quad . \tag{18}$$

For programming purposes we must replace the functional inequalities (18) by a finite set of inequalities. Therefore we select a finite sequence of r points s_k at which to demand that (18) be satisfied.

Further, it proves convenient to replace each inequality by two inequalities so as to eliminate the absolute value signs. Thus we obtain

$$g_k \equiv \left(\sum_{j=1}^{4} \alpha_j n_{jk} \right)^2 + \sum \alpha_j m_{jk} - 1 \leq 0$$

$$h_k \equiv \left(\sum_{j=1}^{4} \alpha_j n_{jk} \right)^2 - \sum \alpha_j m_{jk} - 1 \leq 0, \qquad k = 1,2, \ldots, r \tag{19}$$

where $n_{jk} = n_j(s_k)$, etc. Thus we have formulated the beam problem as the nonlinear programming problem of choosing $\alpha_1, \ldots, \alpha_4$ so as to maximize α_1 subject to the $2r$ inequalities (19).

3. Solution of the Programming Problem

The method used to solve the problem posed in the preceding section is known as the Created-Response Surface Technique and will be referred to as CRST. It was first suggested by Carroll [11] and later given theoretical validation by Fiacco and McCormick [12].

In the CRST method a parameter ζ is introduced and the primal function $P(\alpha, \zeta)$ is defined by

$$P(\alpha, \zeta) = - \alpha_1 - \zeta \sum_{k=1}^{r} (1/g_k + 1/h_k) \qquad (20)$$

where g_k and h_k are defined in (19). The interior of the domain in an α_i space defined by the inequalities (19) is referred to as the feasible domain. If consideration is limited to points in the feasible domain, then it can be shown [12] that for any given ζ, P achieves its minimum value $\overline{P}(\zeta)$ at some point $\overline{\alpha}(\zeta)$ in the feasible domain. Further, if ζ_ℓ is a sequence of values tending to zero, then

$$\lim_{\ell \to \infty} \overline{P}(\zeta_\ell) = - a_0 \qquad (21)$$

Figure 2 shows a schematic flow diagram of the program for automatic computation on the IBM 7090 at Stanford University using the SUBALGOL compiler. Details of the method used for optimizing ζ for a given α_k and of the second-order gradient method used in minimizing $P(\alpha, \zeta)$ arebeyond the scope of this discussion and the interested reader is referred to [12]. However, the method used in manipulation of the mesh size provides an interesting example of interaction between numerical analysis and the physical field of application; it will be described in some detail.

If r points s_k are used, the computation time required will be almost proportional to r, so that there is an obvious advantage to keeping r small. Now, from a physical picture of a curved beam at the yield-point load, we see that four yield hinges will be sufficient to turn the beam into a mechanism. If there is a yield hinge at s_k, then one of the inequalities (19) will be an equality, and at all points where there is not a yield hinge, the strict inequalities will be satisfied. Therefore, for the first coarse mesh we take $r = 4$ and, lacking any better information, we take the points to be equispaced.

Consider now the situation when we have found the solution for this mesh and denote the resultant value α_1 by α^+. If we evaluate the yield inequalities (18) for values of $s \neq s_k$, we may find that they are violated. However, consider a replacement structure whose strength at the points s_k is the same as the given structure, but which is infintely strong for all other values of s; obviously α^+ is the desired yield-point load for the replacement structure, hence it follows from the corollary to the Upper Bound Theorem that the yield-point load α_o of the given structure must satisfy

$$\alpha_o \leq \alpha^+ . \tag{22}$$

Next, using the ultimately fine mesh decided upon, find the value of s for which (18) is violated most severely and replace α_k by $\beta\alpha_k$, $(0 < \beta < 1)$ so that this worst inequality is just an equality. It then follows from the Lower Bound Theorem that

$$\beta \alpha^{+} \leq \alpha_{o} \qquad (23)$$

so that both upper and lower bounds on the desired yield-point load have been obtained.

For the next mesh s_k, we take only those points at which one of the inequalities (18) has a relative maximum* and repeat the reasoning. In applications it turns out that this process need be repeated only once or twice before the bounds (22) and (23) are sufficiently close together to terminate the computation.

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*As a refinement, only that one of (19) which had a relative maximum was retained, and a band of three mesh points was considered for each maximum, thus speeding the convergence.

4. Examples

The purpose of the present paper is to present a method rather than an exhaustive set of calculations, although with the computer program available it is a trivial matter to analyze any beam under any loading. Examples run to test the program included a straight beam under uniform load (a trivial problem designed to discover "bugs" in the program), and a circular arch of arbitrary angle under either a uniform vertical load, a concentrated vertical load at the center, or a uniform load perpendicular to one end of the arch. Figure 3 shows some typical results for the three cases.

5. Extensions

The method described here for the relatively simple problem of a curved beam should prove useful in finding the yield-point loads of more complex two and three dimensional structures. It may be of some interest to mention briefly a two dimensional problem on which research is currently being done:

An annulus of inner and outer radii b and a, respectively is subjected to a uniform uniaxial tensile load F on its outer edge. Assuming conditions of either plane stress or plane strain, the internal equilibrium equations will be satisfied if the three non-vanishing stress components are given in terms of a stress function $\psi(r,\theta)$ by

$$\sigma_r = \psi,_r / r + \psi_{\theta\theta} / r^2$$

$$\sigma_\theta = \psi,_{rr} \qquad \tau_{r\theta} = -(\psi,_\theta / r),_r \tag{24}$$

and the boundary conditions at r=b and r=a will be satisfied if

$$\psi(r,\theta) = (Fa/4)(r-b)^2 / (a-b)\,[1 + \cos 2\theta\,(a^2 + ab - 2br) / (a-b)^2\,]$$

$$+h(r) + \sum_{k=1}^{m} g_m(r) \cos 2m\theta \tag{25}$$

for any well behaved functions h and g_m which satisfy

$$h'(a) = g_m(a) = g_m'(a) = h'(b) = g_m(b) = g_m'(b) = 0\ . \tag{26}$$

The yield inequality will be a quadratic expression in the derivatives of ψ whose precise form will depend upon the material yield condition and upon whether the annulus is in plane strain or plane stress:

$$G[\psi] \leq 0 \qquad (27)$$

The problem is to determine F and the functions $h(r)$ and $g_m(r)$ so as to maximize F subject to (27).

As with the beam problem, (27) is transformed from a functional inequality to a matrix of inequalities by considering a mesh of points r_i, θ_j; presumably similar techniques for keeping the number of mesh points reasonably small can be developed.

An additional complication is the functional form of h and g_m in contrast to the finite vector α_k in the beam problem. Two possibilities are currently being investigated to deal with this phenomenon. On the one hand, h and g_m may be replaced by truncated series of complete functions such as polynomials, trigonometric functions, Bessel functions etc. For example, if

$$h' = \sum_{j=1}^{n} \alpha_{j+1} \sin j\pi(r-b) / (a-b)$$

$$g_m = \sum_{j=1}^{n} (r-b)(a-r)\, \alpha_{mn+j+1} \sin j\pi(r-b) / (a-b) \qquad (28)$$

then by putting the load $F = \alpha_1$ we retain the formalism of the beam problem formulation except that k runs to $m(n+1)+1$ instead of only to 4.

Alternatively, the values of h and g_m at the mesh points may be taken as the unknown α's and the derivatives in (27) replaced by appropriate finite difference formulas.

Still further complications may be introduced if the boundary conditions do not lend themselves to analytic expression. One method of handling this would be to replace a typical boundary condition

$$H[\psi] = 0 \qquad (29)$$

at a point by a pair of boundary inequalities

$$-\epsilon \leq H[\psi] \leq \epsilon \qquad (30)$$

and add the contraints (30) to the yield constraints (27). Alternatively, some positive measure E of the extent to which the required boundary conditions are in error could be calculated, and the primal function formulated as

$$P(\alpha, \zeta) = -F + E - \zeta \sum (1/G[\psi]). \qquad (31)$$

Minimization of P would then lead to the largest load and the smallest boundary error.

Despite the questions which remain to be resolved, it appears likely that the CRST method of nonlinear programming will provide a valuable tool for the calculation of yield-point loads of complex structures.

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APPENDIX

List of Symbols

<u>Beam properties</u>

Typical measurement	A
Width	2B
Height	2H
Yield stress	σ_0
Yield force	$N_o = 4BH\ \sigma_o$
Yield moment	$M_o = 2BH^2\ \sigma_o$

<u>Dimensioned and dimensionless variables and parameters</u>
(Lower case symbols are dimensionless, capitals have dimension)

Axial force	$n = N/N_0$
Shear force	$q = Q/N_o$
Moment	$m = M/M_0$
Normal load	$p_n = P_n A/N_0$
Tangential load	$p_s = P_s A/N_0$
Beam length	$\ell = L/A$
Beam curvature	$\kappa = K A$
Beam constant	$\eta = AN_o/M_0 = 2A/H$
Arc length	$s = S/A$

Mathematical definitions

$n_1(s)$, $m_1(s)$	Particular equilibrium solution under unit load
$n_j(s)$, $m_j(s)$	Complementary equilibrium solutions under zero loads, j = 2, 3, 4
n_{jk}, m_{jk}	$n_j(s_k)$, $m_j(s_k)$ (point values), j = 1, 2, 3, 4
α_1	multiplier of particular solution
α_j	multiplier of complementary solutions, j = 2, 3, 4
α_o	value of α_1 at yield-point load
g_k , h_k	yield functions at $s = s_k$, defined by Eqs. (19)

FIGURE TITLES

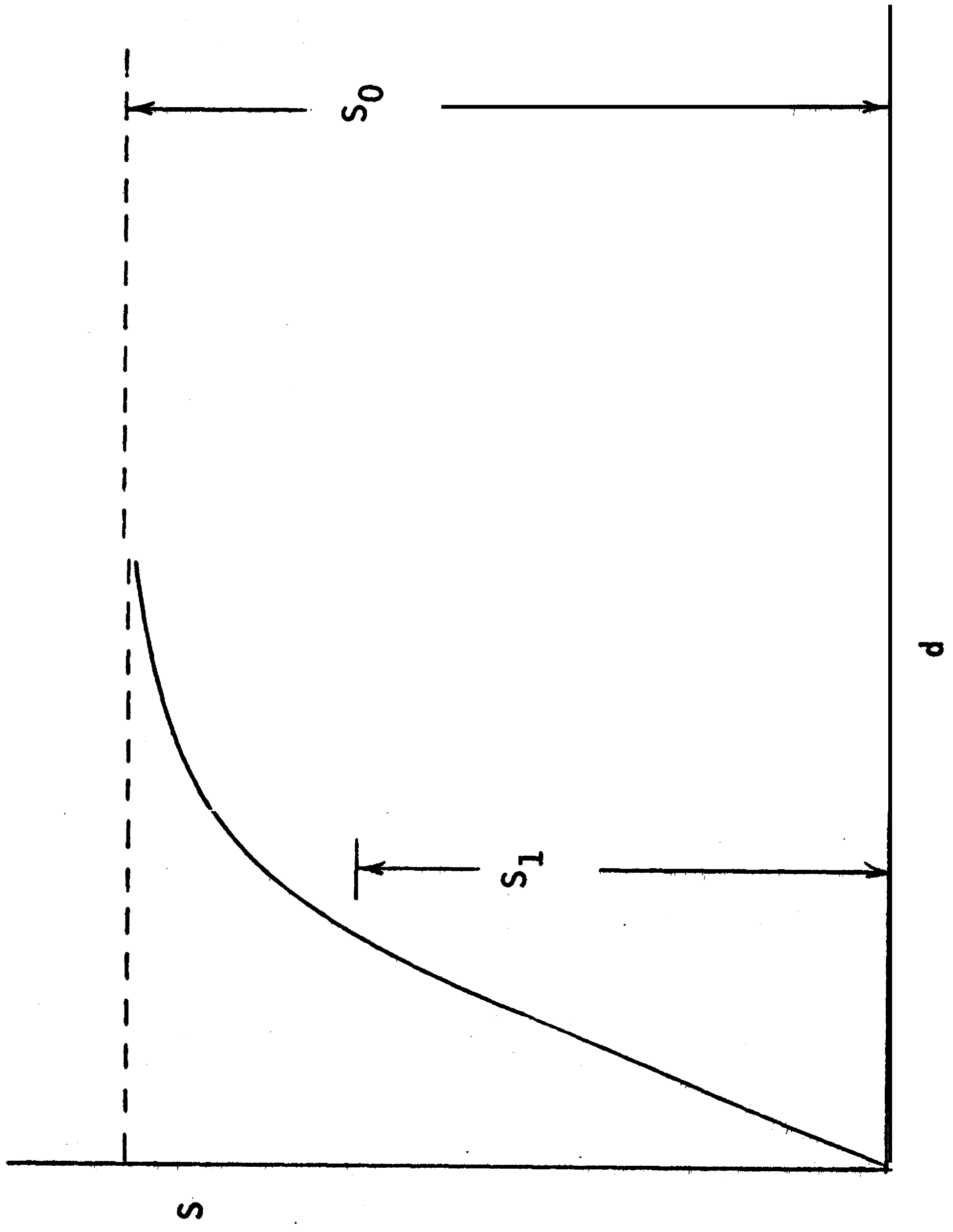

Figure 1 - Typical Load-Displacement Curve

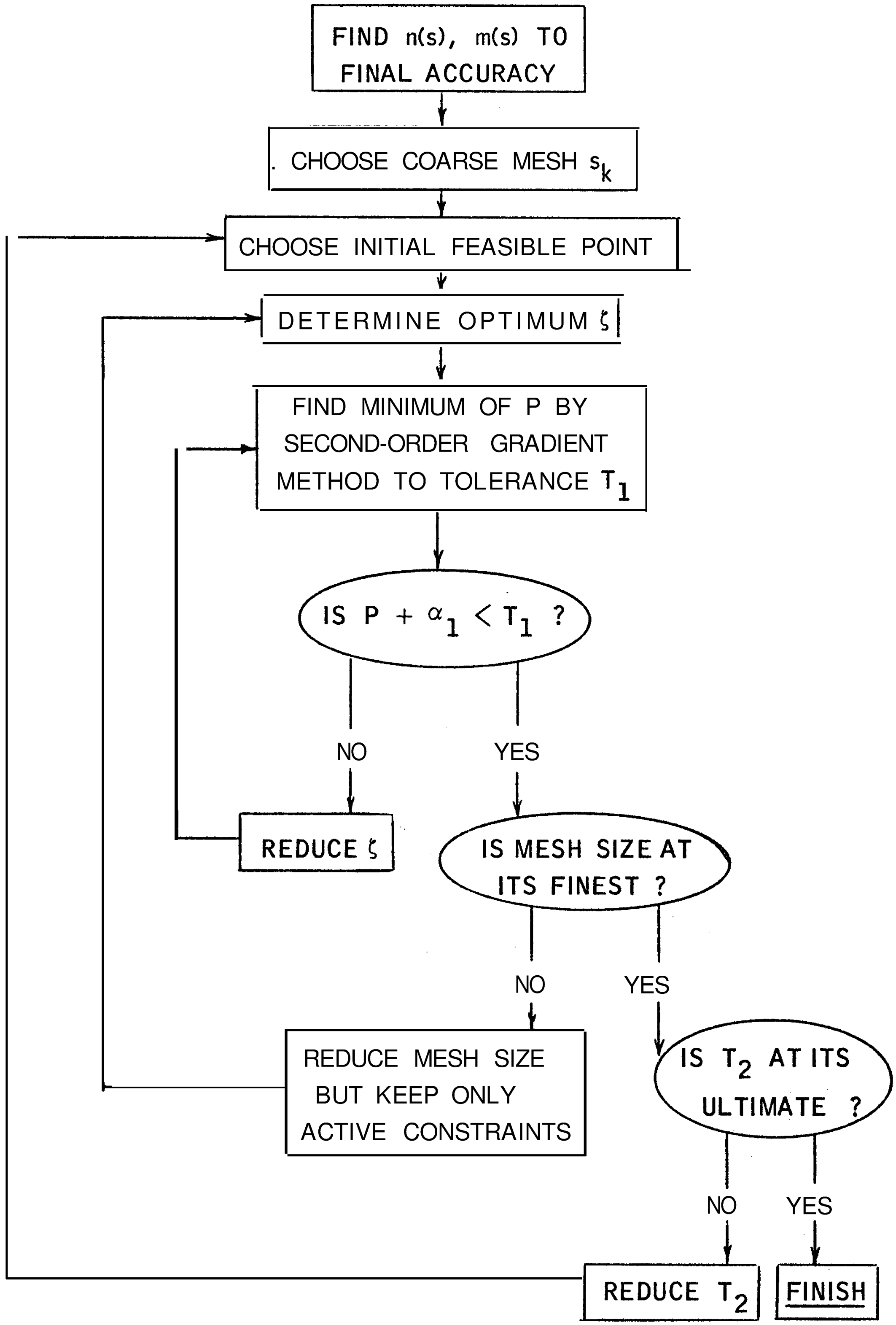

Figure 2 - Schematic Diagram of Computer Program

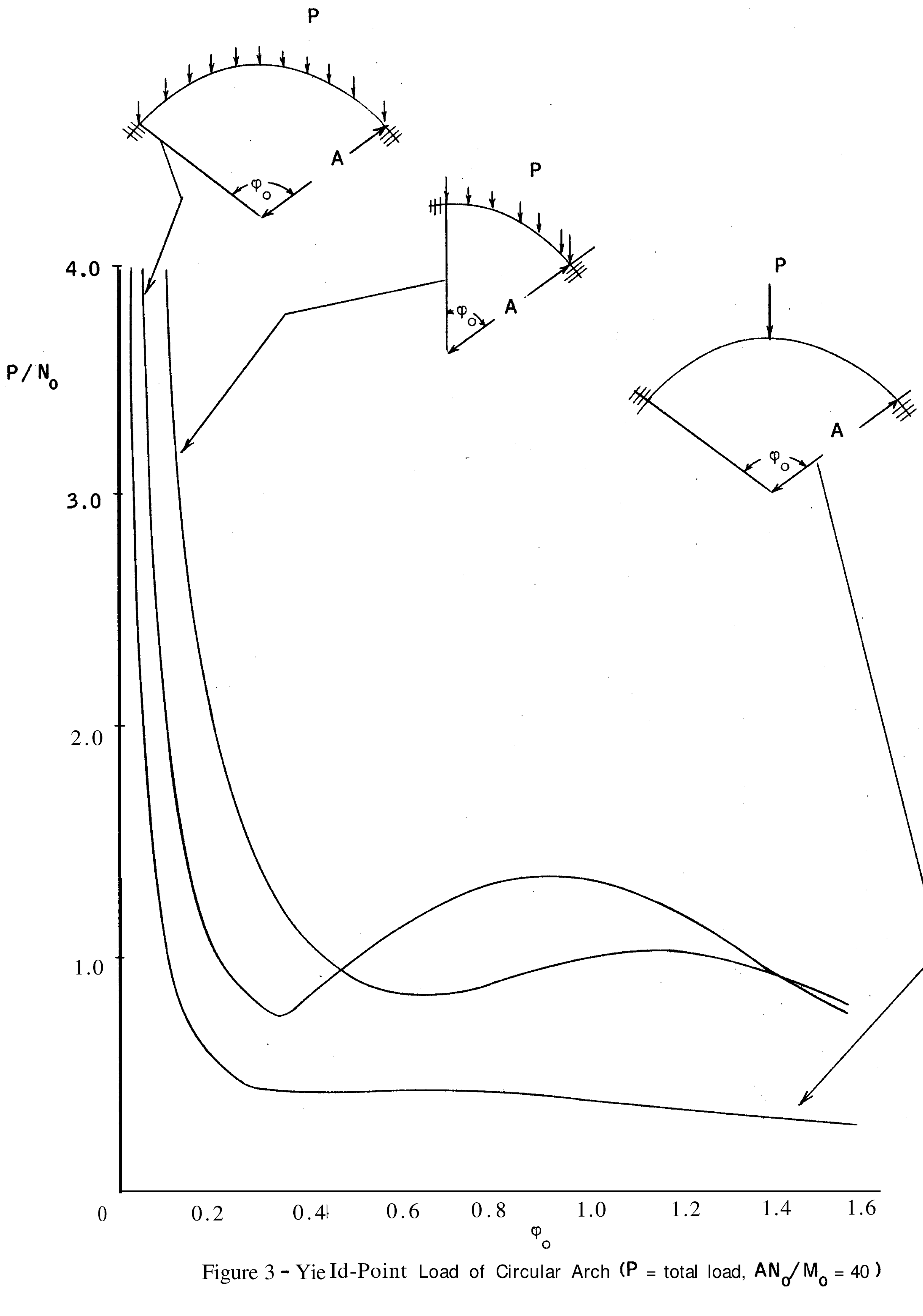

Figure 3 - Yield-Point Load of Circular Arch (P = total load, $AN_0/M_0 = 40$)