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THE MAXIMUM AND MINIMUM OF A POSITIVE DEFINITE QUADRATIC POLYNOMIAL ON A SPHERE ARE CONVEX FUNCTIONS OF THE RADIUS

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Abstract

It is proved that in euclidean n-space the maximum $M(\rho)$ and minimum $m(p)$ of a fixed positive definite quadratic polynomial Q on spheres with fixed center are both convex functions of the radius ρ of the sphere. In the proof, which uses elementary calculus and a result of Forsythe and Golub, $m''(\rho)$ and $M''(\rho)$ are shown to exist and lie in the interval $[2\lambda_1, 2\lambda_n]$, where λ_i are the eigenvalues of the quadratic form of Q. Hence $m''(\rho) > 0$ and $M''(\rho) > 0$.

Summary

Let A be a given symmetric, nonsingular matrix of real elements and order n . Let b be a given column vector of n real elements. For each real column n-vector x , the nonhomogeneous quadratic polynomial

$$Q(x) = (x-b)^T A(x-b)$$

(T denotes transpose) is a real number. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ be the (necessarily) real eigenvalues of A . Let m(ρ) be the minimum of Q(x) on the sphere $S_\rho = \{x: x^T x = \rho^2\}$, and let M(ρ) be the maximum of Q(x) on S_ρ . M. J. D. Powell asked the author whether m(ρ) is a convex function of ρ when A is positive definite. An affirmative answer is given by the theorem:

(1) Theorem. If A is positive definite i.e., if $0 < \lambda_1$) , then both m(ρ) and M(ρ) are convex functions of ρ , for all ρ > 0 .

Theorem (1) will follow from the following result:

(2) Theorem. Let A be any nonsingular matrix. Then for ρ > 0 , the second derivatives m''(ρ) and M''(ρ) both exist, and

(3) $$m''(\rho) \geq 2\lambda_1 \quad \text{and} \quad M''(\rho) \geq 2\lambda_1 .$$

Equality occurs in (3) if and only if $Ab = \lambda_1 b$. Moreover,

(4) $$m''(\rho) \leq 2\lambda_n \quad \text{and} \quad M''(\rho) \leq 2\lambda_n$$

and equality occurs in (4) if and only if $Ab = \lambda_n b$.

Review of Previous Work

The proof of Theorem (2) is based on techniques developed in Forsythe and Golub [1], which dealt only with the case $\rho = 1$. The relevant results of [1] are now summarized and extended to general ρ.

Let $\{u_1,\ldots,u_n\}$ be an orthonormal real set of eigenvectors of A, with $Au_i = \lambda_i u_i (i = 1,\ldots,n)$. Let $b = \sum b_i u_i$. For any vector x in S_ρ at which $Q(x)$ is stationary with respect to S_ρ, there is a real number λ with

$$A(x-b) = \lambda x \tag{5}$$

$$x^T x = \rho^2 \tag{6}$$

Letting $x = \sum x_i u_i$, we find from (5) that

$$x_i = \frac{\lambda_i b_i}{\lambda_i - \lambda} , \tag{7}$$

so that (6) becomes

$$g(\lambda) \equiv \sum_{i=1}^{n} \frac{\lambda_i^2 b_i^2}{(\lambda_i - \lambda)^2} = \rho^2 \tag{8}$$

For each given value of $\rho > 0$, equation (8) determines from 2 to 2n real values of λ. For each λ so determined, equation (5) determines one or more vectors x^λ (if all $b_i \neq 0$, then x^λ is unique). For any x^λ, we have

$$Q(x^\lambda) = f(\lambda) , \tag{9}$$

where

$$(10) \qquad f(h) = \lambda^2 \sum_{i=1}^{n} \frac{\lambda_i b_i^2}{(\lambda_i - \lambda)^2}$$

Now $Q(x)$ is stationary with respect to S_ρ at any x^λ. For given ρ, let $\Lambda_L = \Lambda_L(\rho)$ and $\Lambda_R = \Lambda_R(\rho)$ be the smallest resp. largest values of λ satisfying equation (8). Theorem (4.1) of [1] states that $f(\Lambda_L)$ and $f(\Lambda_R)$ are the minimum resp. maximum values of $Q(x)$ on S_ρ.

Much of [1] was devoted to the singular cases where some $b_i = 0$. For the present investigation, where we are interested only in the values of $Q(x)$, we simply omit from the sums (8) and (10) all terms with $b_i = 0$, and reduce n, if necessary. Having done that, it is then clear from (8) that, for any ρ,

$$(11) \qquad \Lambda_L < \lambda_1 \quad \text{and} \quad A, < \Lambda_R .$$

This concludes the necessary summary of [1].

As a digression, the author notes that the main theorems (2.7) and (4.1) of [1] were proved in [1] by studying $f(\lambda)$ and $g(A)$ for complex values of λ. In late 1965, Professor W. Kahan [unpublished] showed us how to prove those theorems more simply, using only real values of λ.

Proof of Theorem (2).

With the above apparatus our problem is reduced to an exercise in the differential calculus. For each $\rho > 0$ we determine a unique Lagrange multiplier $\lambda = \lambda(\rho)$ from (8) -- either the minimal A_L or maximal A_R. For ease of exposition, suppose $\lambda(\rho) = \Lambda_L$. Then the function

$$(12) \qquad m(\rho) = f(\lambda(\rho))$$

is determined from (10). Since $f(\lambda)$ and g(A) are analytic for $\lambda < \lambda_1$, the function $m(\rho)$ has derivatives of all order. We shall determine m"(p) by calculus. To simplify some expressions, we introduce the abbreviations

$$(13) \qquad \alpha_p = \sum_{i=1}^{n} \frac{\lambda_i^2 b_i^2}{(\lambda_i - \lambda)^2} \qquad (p = 2, 3, 4).$$

Differentiating (10) and simplifying, we find:

$$(14) \qquad \frac{df}{d\lambda} = 2\lambda\alpha_3 \quad ;$$

$$(15) \qquad \frac{d^2f}{d\lambda^2} = 2\alpha_3 + 6\lambda\alpha_4 \quad .$$

Now equation (8) states that, when $\lambda = \lambda(\rho)$,

$$(16) \qquad \alpha_2 = \rho^2 \quad .$$

Differentiating (8) twice with respect to ρ yields

$$(17) \qquad \frac{d\lambda}{d\rho}\alpha_3 = \rho \quad ;$$

$$(18) \qquad \frac{d^2\lambda}{d\rho^2}\alpha_3 + 3\left(\frac{d\lambda}{d\rho}\right)^2 \alpha_4 = 1 \; .$$

Solving (17) and (18) in turn, we find

$$(19) \qquad \frac{d\lambda}{d\rho} = \frac{\rho}{\alpha_3} \quad ;$$

$$(20) \qquad \frac{d^2\lambda}{d\rho^2} = \frac{1}{\alpha_3} - \frac{3\rho^2\alpha_4}{\alpha_3^2}$$

Now, by the chain rule,

$$\frac{dm}{d\rho} = \frac{df}{d\lambda} \cdot \frac{d\lambda}{d\rho} \quad ,$$

and

$$(21) \qquad \frac{d^2m}{d\rho^2} = \frac{d^2f}{d\lambda^2} \left(\frac{d\lambda}{d\rho}\right)^2 + \frac{df}{d\lambda} \cdot \frac{d^2\lambda}{d\rho^2} \quad .$$

We now substitute into (21) the expressions (14), 15), (19), and (20). We find that

$$(22) \qquad m''(\rho) = \frac{d^2m}{d\rho^2} = (2\alpha_3 + 6\lambda\alpha_4)\,\frac{\rho^2}{\alpha_3^2} + 2\lambda\alpha_3\left(\frac{1}{\alpha_3} - \frac{3\rho^2\alpha_4}{\alpha_3^3}\right) \quad .$$

Hence

$$\frac{1}{2}\, m''(\rho) = \lambda + \frac{\rho^2}{\alpha_3} = \frac{1}{\alpha_{-3}}(\lambda\alpha_3 + \alpha_2) \quad , \quad \text{by (16).}$$

Simplifying,

$$\frac{1}{2}\, m''(\rho) = \frac{1}{\alpha_3} \sum_{i=1}^{n} \frac{\lambda_i^3 b_i^2}{(\lambda_i-\lambda)^3} \quad , \quad \text{or}$$

$$(23) \qquad \frac{1}{2}\, m''(\rho) = \sum_{i=1}^{n} \frac{\lambda_i^3 b_i^2}{(\lambda_i-\lambda)^3} \Bigg/ \sum_{i=1}^{n} \frac{\lambda_i^2 b_i^2}{(\lambda_i-\lambda)^3} \quad .$$

Formula (23) is the end of our calculus exercise. In it, λ is determined from solving (8). Note by (11) that the factors $(A_i-A)^3$ all have the same sign for $i = 1, 2, \ldots, n$, whether $\lambda = \Lambda_L$ or $A = \Lambda_R$. Hence $\frac{1}{2}\, m''(\rho)$ is a weighted average with positive weights of the $\{\lambda_i\}$.

It follows that $\frac{1}{2} m''(\rho) \geq \lambda_1$, with equality only when all λ_i in (23) are equal to λ_1 , i.e., if bi = 0 for $\lambda_i > \lambda_1$. This proves (3), and (4) is proved analogously, This concludes the proof of Theorem (2).

It would be desirable to have a simple geometrical proof.

What if A is singular?

If A is singular, that is, if some $\lambda_i = 0$, the situation is somewhat more complicated, just as the case where some $\lambda_1 b_1 = 0$ is complicated in [1]. Theorem (2) fails to hold for semidefinite matrices, because $m''(\rho)$ may not exist for some ρ , as the following example shows:

(24) Example.=' For n = 2 let $Q(x) = (x_2-1)^2$ . . . . . $x = (x_1,x_2)^T$.

Then

$$m(\rho) = \begin{cases} 1-\rho , & 0 \leq \rho \leq 1 , \\ 0 , & 1 \leq \rho < \infty , \end{cases}$$

so $m'(1)$ does not exist.

If $\lambda_1 = 0$, the Lagrange multiplier remains at $\lambda = 0$ for all sufficiently large ρ .

Theorem (1) can easily be extended to semidefinite matrices by continuity. We have

(25) Theorem. If A is positive semidefinite (i.e., if $0 \leq \lambda_1$), then both m(p) and M(p) are convex functions of ρ for $\rho > 0$.

In proof, we note that m(p) and M(p) are continuous functions of the elements of A . If A is semidefinite, it can be approximated by a definite matrix A_ε , for which m_ε and M_ε are convex, with $\|A-A_\varepsilon\| < \varepsilon$. Letting $\varepsilon \to 0$, we find that $m = \lim m_\varepsilon$ and $M = \lim M_\varepsilon$ are convex.

Reference

[1] George E. Forsythe and Gene H. Golub, "On the stationary values of a second-degree -polynomial on the unit sphere", J. Soc. Indust. Appl. Math., vol. 13 (1965), pp. 1050-1068.