ON MULTIPLICATIVE REPRESENTATIONS OF INTEGERS

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Abstract

In 1969 it was shown by P. Erdös that if 0 < al < a2 < . . . < $_{TV--}$ $\le x$ is a sequence of integers for which the products $a_i a_j$ are all distinct then the maximum possible value of k satisfies

$$\pi(x) + c_2\, x^{3/4}/(\log x)^{3/2} < \max k < \pi(x) + c_1\, x^{3/4}/(\log x)^{3/2}$$

where $\pi(x)$ denotes the number of primes not exceeding x and c_1 and c_2 are absolute constants.

In this paper we will be concerned with similar results of the following type. Suppose $0 < a_1 < \dots < a_k < x$, $0 < b_1 < \dots < b_\ell < x$ are sequences of integers. Let g(n) denote the number of representations of n in the form $a_i b_j$. Then we prove:

(i) If $g(n) \le 1$ for all n then for some constant c_3 ,

$$k\ell < \frac{c_3 x^2}{\log x} .$$

(ii) For every c there is an f(c) so that if $g(n) \le c$ for all n then for some constant c_4 ,

$$k\ell < \frac{c_4 x^2}{\log x} (\log \log x)^{f(c)} .$$

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On Multiplicative Representations of Integers

P. Erdös and E. Szemerédi

Let $a_1 < \cdots < a_k \leq x$ be a sequence of integers for which the products $a_i a_j$ are all distinct. P. Erdös proved that [1]

$$\pi(x) + c_2\, x^{3/4}/(\log x)^{3/2} < \max k < \pi(x) + c_1\, x^{3/4}/(\log x)^{3/2} .$$

Perhaps there is an absolute constant c so that

$$(1) \qquad \max k = \pi(x) + c\, x^{3/4}/(\log x)^{3/2} + o\left(\frac{x^{3/4}}{(\log x)^{3/2}}\right)$$

but we can not prove (1). ($c, c_1, \ldots$ denote absolute constants not necessarily the same.)

P. Erdös [2] also proved that if $a_1 < \ldots < a_k \leq x$ is such that the number of solutions of $a_i a_j = t$ is less than $2^{\ell}+1$ then

$$(2) \qquad \max k = (1 + o(1)) \frac{n(\log\log n)^{\ell-1}}{(\ell-1)!\,\log n} .$$

In fact (2) holds if the number of solutions is $< 2^{\ell-1}+1$.

Let $a_1 < a_2 < \ldots$ and denote by $g(n)$ the number of solutions of $n = a_i a_j$. (2) easily implies that if $g(n) > 0$ for all n then $\limsup_{n=\infty} g(n) = \infty$. It is curious to remark that the additive analogues of this result present great difficulties. An old problem of P. Erdös and P. Turán states: Denote by $f(n)$ the number of solutions of $n = a_i + a_j$. Then if $f(n) > 0$ then $\limsup_{n=\infty} f(n) = \infty$. The proof or disproof of this conjecture seems to present surprising difficulties and P. Erdös offered 300 dollars for a proof or disproof.

Raikov proved that if $a_1 < a_2 < \ldots$ is such that $g(n) > 0$ for all n then

$$\limsup_{x=\infty} A(x)\frac{(\log x)^{1/2-\epsilon}}{x} > 0$$

where $A(x) = \sum_{a_i < x} 1$. P. Erdös asked: Is there a sequence $a_1 < a_2 < \ldots$ for which $g(n) > 0$ and $A(x) < cx/\log x$ for infinitely many x ? Wirsing [9] answered this question affirmatively. In fact he showed that $g(n) > 0$ for all $n > n_0$ implies $A(x) > x \frac{x}{\log x}(1+\epsilon)$ for some $\epsilon > 0$ and that this result is best possible; that is, for every $\epsilon > 0$ there is a sequence $a_1 < a_2 < \ldots$ satisfying $g(n) > 0$ for all $n > n_0$ and $A(x) < \frac{x}{\log x}(1+\epsilon)$ for infinitely many x .

Let $1 \leq a_1 < \ldots < a_\ell < x$, $1 \leq b_1 < \ldots < b_\ell < x$. Assume that there are at least cx distinct integers not exceeding x of the form $a_i b_j$. Then $\max(A(x),B(x)) > x^{1/2+d}$ and if the number of distinct $a_i b_i$'s is $x+o(x)$ then $\max(A(x),B(x)) > x^{1-\epsilon}$ for every $\epsilon > 0$.

We do not discuss the proofs here which are not difficult.

It might be worth while to investigate the question that if $g(n) > 0$ and $A(x) < \frac{cx}{\log x}$ holds for infinitely many x, is it then true that $A(x) > cx$ for infinitely many x , or if this would not be true, how fast must A(x) increase for a suitable infinite sequence $x_j \to \infty$.

One more question in this direction: Let $a_1 < \cdots < a_k \le x$ be a sequence of integers for which the products $\prod_{i=1}^{k} a_i^{\varepsilon_i}$, $\varepsilon_i = 0$ or 1 are all distinct. P. Erdös [3] proved

$$\max k = \pi(x) + \pi(x^{1/2}) + O\left(\frac{x^{1/2}}{\log x}\right) .$$

In fact, perhaps the following more precise statement can be made: Let $1 \le u_1 < \cdots < u_k$ be a sequence of integers for which all the sums $\sum_{i=1}^{k} \varepsilon_i u_i$, $\varepsilon_i = 0$ or 1 are all distinct. Put $\min u_k = \alpha_k$. Erdös and Pósa observed that

$$\text{(3)} \qquad \max k \ge \sum_{k=1}^{\infty} \pi\left(x^{1/\alpha_k}\right)$$

and there could be equality in (3). A very old problem of P. Erdös asks: Is it true that $\alpha_k > 2^{k-c}$ for every k where c is an absolute constant? P. Erdös offers 300 dollars for a proof or disproof of this conjecture.

Let $1 \le a_1 < \cdots < a_k < x$; $1 \le b_1 < \cdots < b_\ell \le x$ be two sequences of integers. Assume that all the products $a_i b_j$, $1 \le i \le k$; $1 \le j \le \ell$ are distinct. P. Erdös conjectured and Szemerédi [7] proved that then

$$\text{(4)} \qquad k\ell < \frac{cx^2}{\log x} .$$

First of all we give a simpler proof of (4), which nevertheless uses many of the ideas of the original proof. We conjecture that in fact

(5) $$k\ell \leq (1+o(1))\frac{x^2}{\log x}.$$

It is easy to see that (5) if true is best possible. To see this, let the a's be the primes in $\left(\frac{x}{t}, x\right)$ and the b's are the integers not exceeding x all whose prime factors are $\leq \frac{x}{t}$. Clearly the products $a_i b_j$ are all distinct and the prime number theorem implies

$$k\ell \geq (1+o(1))\frac{x^2}{\log x} \quad \text{if} \quad t = t_x \to \infty \text{ but } t/x^{\varepsilon} \to 0 \text{ for every } \varepsilon > 0 .$$

In fact by choosing $t = \log x\,(1+o(1))$ we maximize $k\ell$ and we then get sequences $a_1 < \ldots < a_k$, $b_1 < \ldots < b_\ell$ with the products $a_i b_j$ all distinct- and

(6) $$k\ell > \frac{x^2}{\log x} - \frac{x^2 \log\log x}{(\log x)^2} + \left(\frac{x^2 \log\log x}{(\log x)^2}\right).$$

It would be of interest to see if (6) can be improved. Conceivably it is best possible, but we have no evidence for it.

In this paper we prove the following theorem. To every c there is an $f(c)$ so that if $1 \leq a_1 < \ldots < a_k \leq x$, $1 \leq b_1 < \ldots < b_\ell < x$ are such that $g(n) < c$ for all n then

(7) $$k\ell < \frac{c_1 x^2}{\log x}(\log\log x)^{f(c)}.$$

(7) is best possible apart from the value of $f(c)$. The proof is not entirely trivial and we only outline it. Let $r > 1$ be given. The sequence B consists of all the squarefree integers b satisfying $\frac{x}{2} < b < x$, and $v(b) \leq r$ ($v(b)$ is the number of prime factors of b). The sequence A consists of all the integers $a < x$ which do not have two divisors $d_1 < d_2 < 2d_1$, $v(d_1) \leq r$, $v(d_2) \leq r$.

It is not difficult to show that

$$A(x) > c_1 x \,, \quad B(x) > c_2 x (\log\log x)^4 / \log x$$

and the number of solutions of $a_i b_j = n$ is less than c_r where c_r depends only on r. We do not discuss the details.

We further outline the proof of the following two theorems:

1. Assume $A(x) > c_1 x$, $B(x) > c_2 x$. Then

$$\max_{n < x^2} g(n) > (\log x)^{c_3} \,. \tag{8}$$

Again apart from the value of c_3 this is best possible. (To see this, let the a's and b's have $\leq \log\log n$ prime factors.) Finally assume $A \cup B$ is the set of all integers and $A(x) > cx$, $B(x) > cx$. Here

$$\max_{n < x^2} g(n) > (\log x)^{c_4 \log\log x} \tag{9}$$

and apart from the value of c_4 this is best possible. To see this, let the a's have $\leq \log\log n$ prime factors and the b's have $> \log\log n$ prime factors. Perhaps (9) holds for every $c_4 < 1-\varepsilon$. The above example shows that it can not hold for $c_4 > 1+\varepsilon$.

Now we are ready to prove (4). In other words we prove the following.

<u>Theorem 1</u>. Let $1 \leq a_1 < \ldots < a_k < x$, $1 \leq b_1 < \ldots < b_\ell < x$ be two sequences of integers. Assume that the products $a_i b_j$ are all distinct. Then for some absolute constant c

$$k\ell < \frac{cx^2}{\log x} \,.$$

Denote by A resp. B the sequences $\{a_1,\ldots,a_k\}$ and $\{b_1,\ldots,b_\ell\}$. $A(y)$ will denote the number of terms of A not exceeding y. A prime p is associated with A if there are at least $\frac{k}{100\, p \log p}$ multiples of p in A, similarly p is associated with B if there are at least $\frac{\ell}{100\, p \log p}$ multiples of p in B. Let $p_1 < p_2 < \cdots$ be the primes which are not associated with A -- omit all the a's which are multiples of any of the p's. Thus we obtain the new sequence A_1 having k_1 terms. Repeat the same process and also apply it to B with the primes not associated with B. Since $\sum_p \frac{1}{100\, p \log p} < \frac{1}{2}$, eventually we obtain a sequence $U = \{u_1 < \ldots < u_{\lambda_1}\}$, $V \subset B$, $\lambda_1 > \frac{k}{2}$ and $V = \{v_1 < \ldots < v_{\lambda_2}\}$, $V \subset B$, $\lambda_2 > \frac{\ell}{2}$ with the property that if $p | u_i$ then p is associated with U and if $p | v_j$ then p is associated with V. To prove our theorem it clearly suffices to show

(10) $$\lambda_1 \lambda_2 < c_1 x^2 / \log x .$$

Let now t with $2^t < x^{1/2}$ be the greatest integer for which there are more than $2^{t/2}$ p's in $(2^t, 2^{t+1})$ which are associated with both U and V. Denote these primes by $p_1,\ldots,p_s$.

(11) $$2^t < p_1 < \cdots < p_s < 2^{t+1}, \quad s > 2^{t/2} .$$

Consider the set of all pairs of integers

(12) $$\left\{ \frac{u_j}{p_i}, \frac{v_j}{p_i} \right\}, \quad 1 \le i \le s$$

where $u_j \equiv 0 \pmod{p_i}$, $v_{j'} \equiv 0 \pmod{p_i}$. Since p_i is associated with both U and V there are by (11) at least

$$(13) \qquad \frac{c\lambda_1\lambda_2}{10^4(t+1)^2\, 2^{2t+2}} > \frac{c\lambda_1\lambda_2}{t^2\, 2^{3t/2}}$$

pairs (12).

Now observe that the pairs (12) are unique. If

$$\alpha = \frac{u_{j_1}}{p_{i_1}} = \frac{u_{j_2}}{p_{j_2}} \quad \text{and} \quad \beta = \frac{v_{j_1'}}{p_{i_1}} = \frac{v_{j_2'}}{p_{j2}}$$

then $u_{j1} v_{j_2'} = u_{j_2} v_{j_1'} = \alpha\beta p_{i_1} p_{i_2}$ which contradicts our assumptions.

Now we estimate the number of pairs (12) from above. Denote by $2^{t+1} < p_1 < \cdots < x$ the primes associated with both U and V. By the maximality of t there are at most $2^{\ell/2}$ primes p in the interval $(2^\ell, 2^{\ell+1})$ for every $\ell > t$. Thus trivially

$$(14) \qquad \sum_i \frac{1}{p_i} < 8 \quad .$$

Denote by $q_1 < \cdots$ the primes in $(2^{t+1}, x)$ not associated with U and by $r_1 < \cdots$ the primes in $(2^{t+1}, x)$ not associated with V. Clearly the integers (12) satisfy

$$(15) \qquad \frac{u_j}{p_i} < \frac{x}{2^t}, \quad \frac{v_{j'}}{p_i} < \frac{x}{2^t} \quad \text{and} \quad \frac{u_j}{p_i} \not\equiv 0 \pmod{q}, \quad \frac{v_{j'}}{p_i} \equiv 0 \pmod{r}$$

for all the primes q and r defined above. By Brun's method we immediately obtain from (15) that the number of integers of the form $\frac{u_j}{p_i}$ is less than

(16) $$c_1 \frac{x}{2^t} \prod \left(1 - \frac{1}{q_i} \right)$$

and the number of integers of the form $p_i^{v_j'}$ is less than

(17) $$c_2 \frac{x}{2^t} \prod \left(1 - \frac{1}{r_i} \right)$$

Thus, from (16) and (17) we obtain that the number of pairs (12) is less than

(18) $$c_1 \, c_2 \quad \frac{x^2}{2^{2t}} \prod \left(1 - \frac{1}{q_i} \right) \prod \left(1 - \frac{1}{r_i} \right) \quad .$$

From (14) and the theorem of **Mertens** we obtain

(19) $$\sum \frac{1}{q_i} + \sum \frac{1}{r_i} = \sum_1 \frac{1}{p} - \sum \frac{1}{p_i} > \log \log x - \log t - c$$

where in $\sum_1$ the summation is extended over all the primes in $(2^{t+1}, x)$. **From** (18) and (19) we obtain that the number of pairs (12) is less than

(20) $$c_3 \frac{x^2 t}{2^{2t} \log x} \quad .$$

From (13), (20) and the uniqueness of the pairs (12) we thus obtain

$$\frac{c \lambda_1 \lambda_2}{t^2 \, 2^{3t/2}} < \frac{c_3 \, x^2 t}{2^{2t} \log x}$$

or

$$\lambda_1 \lambda_2 < c \frac{x^2 t^3}{2^{t/2} \log x}$$

which proves (10) and **completes** the proof of Theorem 1.

Observe that if no t exists for which there are many primes in $(2^t, 2^{t+1})$ associated with both U and V , the proof gives

$$U(x)V(x) < c x^2/\log x .$$

If there is a large t then in fact $U(x)V(x) = o\left(\frac{x^2}{\log x}\right)$.

Now let us try to obtain $A(x)B(x) < (1+o(1)) \frac{x^2}{\log x}$. One can formulate this as an extremal problem in number theory. Assume $1 \leq a_1 < \cdots < a_k < x$, $1 \leq b_1 < \ldots < b_\ell < x$ are such that the products $a_i b_j$ are all distinct. What is the maximum of $k\ell$ and which sequences realize this maximum? Perhaps the sequence defined in the introduction comes close but we have no evidence. One could try first of all to prove that the extremal sequence has the following structure: Split the primes into two classes q_i and r_j. The A's are the integers composed of the q's and the B's are the integers composed of the r's . We have not been able to show this -- the method which we use in proving Theorem 1 shows that we can assume that the extremal sequence has the following structure: The primes are split into three classes $\{q_i\}$, $\{r_j\}$, $\{s_\ell\}$ $\sum \frac{1}{s_\ell} < C$ and all the q's are associated with A , all the r's with B and the s's can be associated with both.

If we would succeed in eliminating the primes s then to prove $A(x)B(x) < (1+o(1)) \frac{x^2}{\log}$ we would need the following theorem on sieves which we can not prove but which perhaps can be attacked by the experts: Let $q_1 < q_2 < \ldots$, $r_1 < r_2 < \ldots$ be two disjoint sequences of primes. $a_1 < a_2 < \ldots$, $b_1 < b_2 < \ldots$ are the integers composed of the q's and r's respectively. Is it true that

$$A(x)B(x) \le (1+ o(1)) \frac{x^2}{\log x} \quad (21)$$

As shown in the introduction, equality is possible in (21), but perhaps the only way to achieve equality in (21) is to have $\min\left(\sum \frac{1}{q_i}, \sum \frac{1}{r_i}\right)$ tend to 0 as $x \to \infty$.

Theorem 2. Let $A(x) > c_1 x$, $B(x) > c_2 x$. Then for some $n < x$, $g(n) > (\log x)^\alpha$.

Theorem 2 is an immediate consequence of an old theorem of Erdös [4]. The number of products of the form $a_i b_j$ is $> c_1^2 x^2$ but there are fewer than $\frac{x^2}{(\log x)^\alpha}$ distinct integers of the form $k\ell$, $k = x$, $\ell < x$. This implies Theorem 2.

It would be interesting to determine the best possible value of a, $\alpha \le \frac{1}{\log 2}$ is easy to prove, and at present it is not clear to us how much this can be improved.

Theorem 3. Let $A(x) > cx$, $B(x) > cx$ and assume that every $m < x$ is either in A or B. Then for some $n < x$ and $x > x_0(\varepsilon)$,

$$g(n) > (\log x)^{(\frac{1}{4} - \varepsilon)\log\log x} \quad (22)$$

Denote by I the interval

$$\left(c(\log x)^\eta, c(\log x)^{1/2}\right)$$

and let $p_1 < \ldots < p_s$ be-the primes in I.

$$k = \left[\frac{1}{2}(\log x)^{1/2}\right] , \ \ell = \sum_{i=1}^{s} \frac{1}{p_i} = \left(\frac{1}{2} - \eta\right) \log\log x + O(1) \quad .$$

Denote by D the sequence $d_1 < d_2 < \ldots$ of integers not exceeding x which have at least k distinct prime factors in I . It is easy to see that

$$(23) \qquad D(x) > \frac{x}{2\log x} \ell^{k-1}/(k-1)!$$

The proof of (23) follows the method of Hardy and Ramanujan [6] and will be suppressed.

Without loss of generality we can assume that at least $\frac{1}{2}D(x)$ of the d's are in A (since $A \cup B$ contains all the integers not exceeding x).

It follows from Turán's method [8] that all but $o(x)$ integers not exceeding x have $\ell + o(\ell)$ distinct prime factors in I . Thus since $B(x) > cx$ we can assume that at least $\frac{cx}{2}$ of the b 's have at least t distinct prime factors in I where $t = [(1-\varepsilon)\ell]$. Consider now all the integers

$$(24) \qquad a_i b_j \ , \quad a_i \in D \cap A \ , \quad v(b_j) \geq t \quad .$$

By (24) the number of these products is greater than

$$(25) \qquad \frac{x^2}{(\log x)^2} \ell^{k-1}/(k-1)! \qquad .$$

It is not difficult to see that almost all of these products are squarefree and these then have at least $k+\ell$ prime factors in 1 . It is easy to see that the number of integers not exceeding x which have at least $k+\ell$ distinct prime factors in I is less than

(26) $$x\left(\sum_{i=1}^{s}\frac{1}{p_i}\right)^{k+\ell-1}\Big/(k+\ell-1)! = x\ell^{k+\ell-1}/(k+\ell-1)! \quad .$$

From (25) and (26) we obtain that there is an n for which the number of solutions of $n = a_i b_j$ is at least

$$\frac{x}{(\log x)^2}\ \ell^{k-1}/(k-1)!\left(\frac{x\ell^{k+\ell-1}}{(k+\ell-1)!}\right)^{-1} > \frac{k^\ell}{\ell^\ell} > (\log x)^{(\frac{1}{4}-\varepsilon)\log\log x}$$

which proves (21).

Perhaps (21) holds with $1-\varepsilon$ instead of $\frac{1}{4}-\varepsilon$. To make the proof work, I would have to be the interval

$$\left(c^{(\log x)^{\eta}},\ c^{(\log x)^{1-\eta}}\right),\ k = [(\log x)^{1-\eta}] \quad .$$

But then we could not prove (23), but we hope to return to this question.

Finally we prove

Theorem 4. To every c there is an f(c) so that if $1 \le a_1 < \ldots < a_k < x$, $1 \le b_1 < \ldots < b_\ell \le x$ are such that $g(n) < c$ then (7) holds.

For simplicity we only prove this for $c = 4$. Assume that

(27) $$k\ell > \frac{x}{\log x}(\log\log x)^{\alpha}$$

where α is sufficiently large. We are going to prove that (27) implies that there are integers z, y and four primes $p_1^{(0)}$, $p_1^{(1)}$, $p_2^{(0)}$, $p_2^{(1)}$ so that for all choices of $\varepsilon_i = 0$ or 1, $i = 1,2$,

(28) $$y\prod_{i=1}^{2} p_i^{(\varepsilon_i)} \in A\ ,\quad z\prod_{i=1}^{2} p_i^{(\varepsilon_i)} \in B \quad .$$

(28) clearly implies that $g(z\,y\,p_1^{(0)}p_1^{(1)}p_2^{(0)}p_2^{(1)}) \geq 4$, hence to prove Theorem 4 it suffices to prove (28). In view of the fact that we do not try to get best possible values of α the proof of this will in some respect be simpler than the proof of Theorem 1.

We say that the prime p belongs to A if there are at least $\frac{k}{p(\log\log p)^2}$ multiples of it in A . This is a slight modification of the definition in Theorem 1 (which as the attentive reader will later see is really needed here) but since $\sum \frac{1}{p(\log\log p)^2}$ converges this makes no difference.

Let t_1 be the smallest integer satisfying

$$(\log x)^c < 2^{t_1} < x^{1/2} \quad \text{(where c is sufficiently large)}$$

for which there are more than $\frac{2^{t_1}}{t_1(\log t_1)^2}$ primes which belong to both A and B . If no such interval exist then Brun's sieve gives as in the proof of Theorem 1 that

$$k\ell < \frac{cx^2 \log\log x}{\log x}$$

which implies that in this case our theorem holds.

Let now

$$(29) \qquad p_1, p_2, \ldots, p_s \;, \quad s > \frac{2^{t_1}}{t_1(\log t_1)^2}$$

be the primes in $\left(2^{t_1}, 2^{t_1+1}\right)$ which belong to both A and B . Denote by A_{p_i} respectively B_{p_i} the set of integers

$, \left\{ \frac{a_j}{p_i} \right\}, \left\{ \frac{b_{j'}}{p_i} \right\}, a_j \equiv 0 \pmod{p_i}, b_{j'} \equiv 0 \pmod{p_i}$.

Let now $t_2^{(i)}$ be the smallest integer satisfying

$$(\log x)^c < 2^{t_2^{(i)}} < \left(\frac{x}{p_i}\right)^{1/2}$$

for which there are more than $\frac{2^{t_2^{(i)}}}{t_2^{(i)}(\log t_2^{(i)})^2}$ primes $p_j^{(i)}$ in $\left(2^{t_2^{(i)}}, 2^{t_2^{(i)}+1}\right)$ which belong to both A_{p_i} and B_{p_i}. If such a $t_2^{(i)}$ does not exist then every prime q in $\left((\log x)^c, \frac{x}{p_i}\right)^{1/2}$ belongs to at most one of the sequences A_{p_i}, B_{p_i} (we neglected a set of primes the sum of whose reciprocals goes to 0 as $x \to \infty$ and which may belong to both A and B). But then as in the proof of Theorem 1 we obtain by Brun's method

$$(30) \qquad |A_{p_i}|\,|B_{p_i}| < \frac{c\,x^2 \log\log x}{p_i^2 \log x}$$

Thus from (30) and the definition of A_{p_i}, B_{p_i} we have

$$k\ell = |A|\,|B| \le |A_{p_i}|\,|B_{p_i}|\,p_i^2 (\log p_i \log\log p_i)^4 < \frac{c\,x^2 (\log\log x)^5}{\log x}$$

which again proves Theorem 4.

The number of possible choices for $t_2^{(i)}$ is at most $\log x$ and so there are at least

$$\frac{2^{t_1}}{t_1(\log t_1)^2 \log x}$$

primes p_i in $\left(2^{t_1}, 2^{t_1+1}\right)$ which have the same t_2.

Let p_i, $1 \leq i \leq s$, be the primes (28) and $q_1, \ldots$ the set of primes in $\left(2^{t_2}, 2^{t_2+1}\right)$. To every p_i there are at least

$$\frac{2^{t_2}}{t_2(\log t_2)^2}$$

$p_j^{(i)}$'s (which are q's) so that there are at least

$$\frac{ck}{p_i p_j^{(i)} (\log\log p_i)^2 (\log\log p_j^{(i)})^2} > \frac{ck}{2^{t_1+t_2}(\log t_1)^2 (\log t_2)^2}$$

$$> \frac{cx}{2^{t_1+t_2}(\log t_1)^2 (\log t_2)^2 \log x} \qquad \left(\text{since } k > \frac{x}{\log x}\right)$$

integers $u \leq \dfrac{x}{p_i p_j^{(i)}}$ so that $u p_i p_j^{(i)} \in A$. Therefore by a simple computation there is an integer U to which there are at least

$$\frac{2^{t_1+t_2}}{(\log x)^3}$$

products $p_i p_j^{(i)}$ for which $U p_i p_j^{(i)} \in A$. Henceforth we only consider these pairs $p_i p_j^{(i)}$ which belong to U. To each of these pairs there are at least

$$\frac{cx}{\log x \; 2^{t_1+t_2} (\log t_1)^2 \; (\log t_2)^2}$$

-integers $v < \frac{x}{p_i p_j^{(i)}}$ so that $v p_i p_j^{(i)} \in B$. Thus again by a simple averaging process there is a V so that there are at least

$$\frac{2^{t_1+t_2}}{(\log x)^5}$$

pairs $p \cdot q$ for which $Upq \in A$, $Vpq \in B$.

Now we use the following simple lemma on graphs. Let G be a bipartite graph of L_1 white and L_2 black vertices and more than

$$L_1^{1/2} L_2$$

edges $(L_1 < L_2)$. Then the graph contains a rectangle. Since $2^{t_1} > (\log x)^{100}$, $2^{t_2} > (\log x)^{100}$ the lemma applies and the rectangle gives the configuration which we require.

For $c = 2^k$ the proof is similar. We have to apply our procedure k times and have to use the theorem on k-tuples in [5].

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