SOME MONOTONICITY PROPERTIES OF PARTIAL ORDERS

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Some Monotonicity Properties of Partial Orders

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Abstract.

A fundamental quantity which arises in the sorting of n numbers $a_1, a_2, \ldots, a_n$ is $\Pr(a_i < a_j \mid P)$, the probability that $a_i < a_j$ assuming that all linear extensions of the partial order P are equally likely. In this paper we establish various properties of $\Pr(a_i < a_j \mid P)$ and related quantities. In particular, it is shown that $\Pr(a_i < b_j \mid P') \geq \Pr(a_i < b_j \mid P)$, if the partial order P consists of two disjoint linearly ordered sets $A = \{a_1 < a_2 < \ldots < a_m\}$, $B = \{b_1 < b_2 < \ldots < b_n\}$ and $P' = P \cup \{\text{any relations of the form } a_k < b_\ell\}$. These inequalities have applications in determining the complexity of certain sorting-like computations.

Keywords. Boolean lattices, complexity, Hall's theorem, linear extensions, monotonicity, partial order, probability, sorting.

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1. Introduction.

Many algorithms for sorting n numbers $\{a_1, a_2, \ldots, a_n\}$ proceed by using binary comparisons $a_i : a_j$ to build successively stronger partial orders P on $\{a_i\}$ until a linear order emerges (see, e.g. Knuth [3]). A fundamental quantity in deciding the expected efficiency of such algorithms is $\Pr(a_i < a_j \mid P)$, the probability that the result of $a_i : a_j$ is $a_i < a_j$ when all linear orders consistent with P are equally likely. In this paper we prove some intuitive but nontrivial properties of $\Pr(a_i < a_j \mid P)$ and related quantities. These results are important, for example, in establishing the complexity of selecting the k-th largest number [7].

We begin with a motivating example. Suppose that tennis skill can be represented by a number, so that player x will lose to player y in a tennis match if $x < y$. Imagine a contest between two teams $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ where within each team the players are already ranked as $a_1 < a_2 < \ldots < a_m$ and $b_1 < b_2 < \ldots < b_n$. If the first match of the contest is between a_1 and b_1, what is the probability p that a_1 will win? Supposing that the two teams have never met before, it is reasonable to assume that all relative rankings among players of $A \cup B$ are equally likely, provided they are consistent with $a_1 < a_2 < \ldots < a_m$ and $b_1 < b_2 < \ldots < b_n$. It is easy to show by a simple calculation that $p = m/(m+n)$. Consider now a different situation when the two teams did compete before with results $a_{i_1} < b_{j_1}, a_{i_2} < b_{j_2}, \ldots, a_{i_t} < b_{j_t}$; in other words, the team B players always won. Let p' be the probability for $a_1 < b_1$ assuming that all orderings of elements in $A \cup B$ consistent with the known constraints are equally likely. One would certainly expect that $p' \geq p$, as the additional information indicates that the players on team B are better than those on team A. However, the proof of this does not seem to be so trivial. The purpose of this paper is to establish several general theorems concerning such monotone properties.

We now give a proof[*] that $p' > p$ in the preceding example. It establishes the result even when A and B are themselves only partially ordered, provided that a_1 and b_1 are the unique minimum elements in A and B, respectively. Let us denote by P' the partial order obtained by adding the relations $\{a_{i_1} < b_{j_1}, a_{i_2} < b_{j_2}, \ldots, a_{i_t} < b_{j_t}\}$ to $P = A \cup B$. We will show that $\Pr(a_1 < b_1 \mid P') / \Pr(b_1 < a_1 \mid P') \geq m/n$, from which it follows that $\Pr(a_1 < b_1 \mid P') > m/(m+n) = \Pr(a_1 < b_1 \mid P)$.

Consider the sets S_0 of all $\frac{(m+n-1)!}{(m-1)!(n-1!}$ possible sequences of 0's and 1's with one element "underlined", where

(i) the sequence is of length $m+n$, with m 0's and n 1's,

(ii) the first character is 0,

(iii) one of the 1's is underlined.

Define the set S_1 similarly but with first character 1 and with one of the 0's underlined. We get a 1-1 correspondence between S_0 and S_1 by complementing both the first character and the underlined character. If $x_0 \in S_0$ corresponds to $x_1 \in S_1$, then $x_0 < x_1$ in the partial order $<$ defined on (0,1)-sequences as follows: Say that $x < y$ if we can transform x into y by one or more replacements of '01' by '10'; or, equivalently, $x < y$ if x and y have the same number of 0's, and

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[*] The proof given here is due to D. Knuth.

for all k the position of the k-th 0 of x is no further to the right than the k-th 0 of y .) List all the pairs of the correspondence as $x_0 \leftrightarrow x_1$, $y_0 \leftrightarrow y_1$, ...

For a partial order Q on a set X , we say that a 1-1 mapping $\lambda: X \to \{1,2,\ldots,n\}$ is a linear extension of Q if $\lambda(x) < \lambda(y)$ whenever $x < y$ in Q . Let λ_{x_1} be a linear extension of P' which places elements of A into the positions where x_1 has a 0 , and elements of B into the positions where x_1 has a 1 . The correspondence $x_0 \leftrightarrow x_1$ naturally associates to λ_{x_1} a linear extension λ_{x_0} of P' in which the relative order of the a_i and also the relative order of the b_j are both unchanged. We therefore obtain a list of inequalities $N(x_1) \le N(x_0)$, $N(y_1) \le N(y_0)$, . . . , where $N(x_i)$ denotes the number of all linear extensions λ_{x_i} defined above. (For some x_i , $N(x_i)$ may be 0 .) Summing all the inequalities gives

$$m\cdot(\#\text{ of linear extensions of } P' \cup (b_1 < a_1)$$

$$\le n\cdot(\#\text{ of linear extensions of } P' \cup (a_1 < b_1)\ ,$$

which is what we wanted to show.

The preceding example suggests the following conjecture. Let $A = \{a_1,a_2,\ldots,a_m\}$, $B = \{b_1,b_2,\ldots,b_n\}$, $X = A \cup B$, and $(P,<)$ be a partial order on X which contradicts no relation of the form $b_j < a_i$ (see Figure 1).

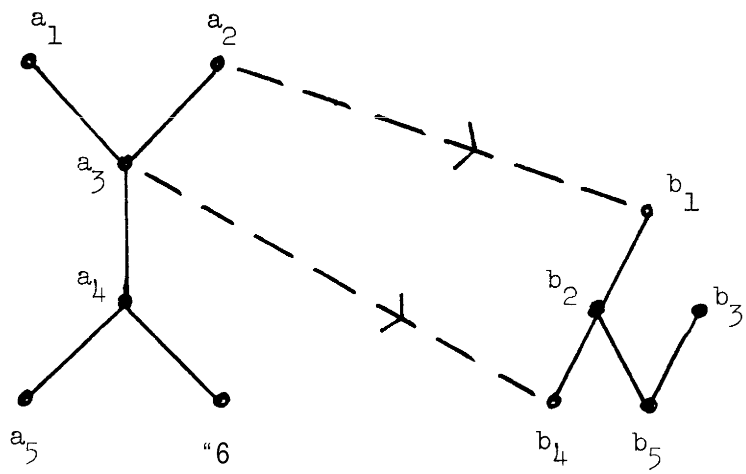

Figure 1. A partial order P generated by $A \cup B \cup \{a_2 < b_1, a_3 < b_4\}$;

P contains no relation of the form $b_j < a_i$.

Conjecture. If P' is a partial order obtained from P by adding relations of the form $a_k < b_\ell$, then $\Pr(E \mid P') \geq \Pr(E \mid P)$, where E is any event of the form $(a_{i_1} < b_{j_1}) \wedge (a_{i_2} < b_{j_2}) \wedge \dots \wedge (a_{i_t} < b_{j_t})$.

In this paper we shall prove several results related to this conjecture, which in particular implies the conjecture for the case when both A and B are linear ordered under P (see Corollary 2 to Theorem 1). The general conjecture, however, remains unresolved.

2. A Monotonicity Theorem.

In this section we shall **prove** a theorem which implies an important special case of the Conjecture, namely, the case when A and B are each linearly ordered under P. In fact in this case the Conjecture is true even if P includes relations of both of the types $a_i < b_j$ and $b_k < a_\ell$.

Let $A = \{a_1 < a_2 < \dots < a_m\}$ and $B = \{b_1 < b_2 < \dots < b_n\}$ be linear orders. Let Λ denote the set of all linear extensions of $P = A \cup B$. A cross-relation between A and B is a set $Z \subseteq (A \times B) \cup (B \times A)$, interpreted as a set of comparisons $a_i < b_j$ and $b_k < a_\ell$. For a cross-relation Z, we define $\hat{Z} = \{\lambda \in \Lambda : \lambda(x) < \lambda(y) \text{ for all } (x,y) \in Z\}$.

It will be convenient to represent each $\lambda \in \hat{Z}$ as a lattice path $\bar{\lambda}$ in $\mathbb{Z}^2$ starting from the origin and terminating at the point (n,m) (see Figure 2). The interpretation is as follows: As we step along λ starting from $(0,0)$, if the k-th step increases the A (or B) coordinate from $i-1$ to i then λ maps a_i (or b_i, **respectively)** to k. Thus, in Figure 2, $\lambda(a_1) = 1$, $\lambda(b_1) = 2$, $\lambda(b_2) = 3$, $\lambda(a_2) = 4$, etc.

Let us consider the geometrical implications of a constraint of the form $\lambda(a_i) < \lambda(b_j)$. By definition, as we go along $\bar{\lambda}$ from $(0,0)$ to (n,m), $\bar{\lambda}$ must achieve an A-value of i before it achieves a B-value of j. But this means exactly that $\bar{\lambda}$ must not pass through the (closed) vertical line segment joining (j,i) to $(j,0)$. In general, a set $X \subseteq A \times B$ represents a set of vertical "barriers" of this type which for any $\lambda \in \hat{X}$, the corresponding lattice path $\bar{\lambda}$ is prohibited from crossing (Figure 3). Of course, a set $Y \subseteq B \times A$ corresponds to a set of horizontal barriers in a similar way, with $(b_j, a_i) \in Y$ being

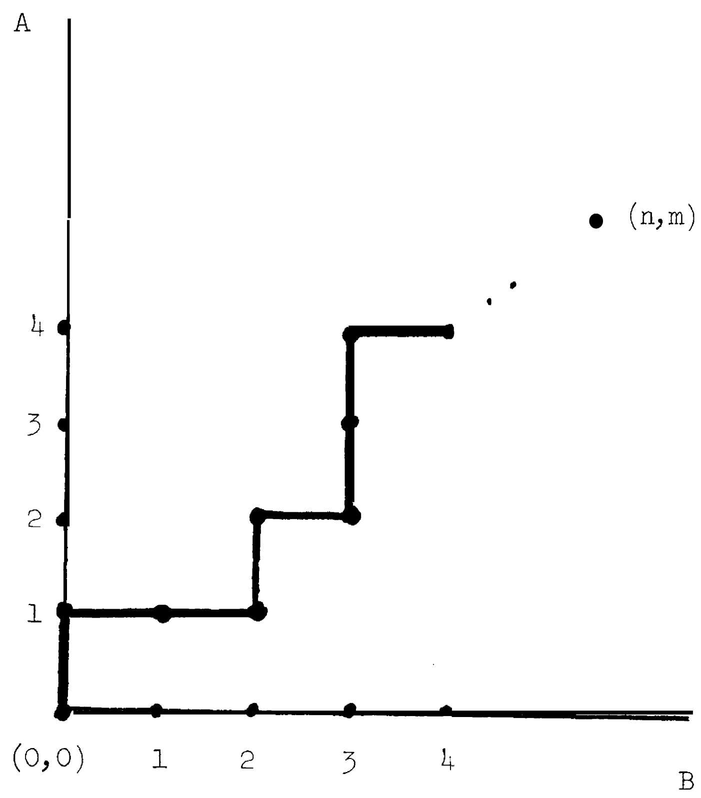

Figure 2.

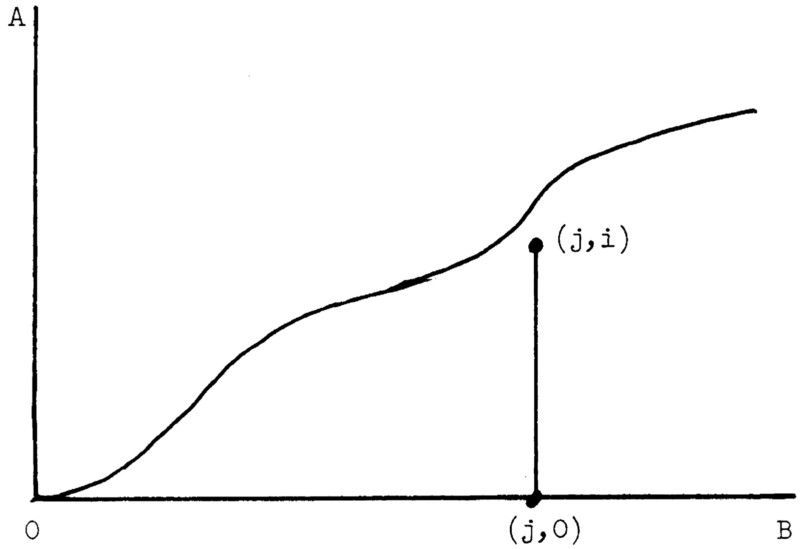

Figure 3. A vertical barrier corresponding to the condition $\lambda(a_i) < \lambda(b_j)$.

represented by the line segment joining $(0,i)$ to (j,i). We will also refer to such vertical and horizontal barriers as x-barriers and y-barriers. For a cross-relation $Z \subseteq (A \times B) \cup (B \times A)$, we define $Z_X = Z \cap (A \times B)$ and $Z_Y = Z \cap (B \times A)$. Thus Z_X and Z_Y are the vertical and the horizontal barriers determined by Z, respectively.

Let Z and W be two cross-relations between A and B. We say Z is more <u>A-selective</u> than W if both $W_X \subseteq Z_X$ and $Z_Y \subset W_Y$. (For example, a set of x-barriers is always more A-selective than a set of y-barriers.) Intuitively, one would think that in this case linear extensions of Z should have a greater probability for ranking A's elements below B's. Let Z' and W' be another pair of cross-relations with Z' being more A-selective than W'. The basic result we prove is the following:

<u>Theorem 1</u>. $|\hat{Z} \cap \hat{Z}'| \cdot |\hat{W} \cap \hat{W}'| \geq |\hat{Z}' \cap \hat{W}| \cdot |\hat{Z} \cap \hat{W}'|$.

<u>Corollary 1</u>. $\dfrac{\Pr(Z' \mid Z)}{\Pr(W' \mid Z)} \geq \dfrac{\Pr(Z' \mid W)}{\Pr(W' \mid W)}$ when the denominators are not zero.

Corollary 1 follows immediately from Theorem 1. It asserts that the ratio $\Pr(Z')/\Pr(W')$ is larger when conditioned on Z than when conditioned on W.

<u>Corollary 2</u>. $\Pr(V \mid Z) \geq \Pr(V \mid W)$ for any V with $V_Y \subset Z_Y$. In particular, $\Pr(X \mid Z) \geq \Pr(X \mid W)$ for any $X \subseteq A \times B$.

This follows from Corollary 1 by letting $Z' = V$, and choosing W' so that $W'_X = \emptyset$ and $W'_Y = V_Y$.

Proof of Theorem 1. We will construct a 1-1 mapping of $(\hat{Z}' \cap \hat{W}) \times (\hat{Z} \cap \hat{W}')$ into $(\hat{Z} \cap \hat{Z}') \times (\hat{W} \cap \hat{W}')$. Suppose $\lambda \in \hat{Z}' \cap \hat{W}$ and $\lambda' \in \hat{Z} \cap \hat{W}'$. Let $\bar{\lambda}$, $\bar{\lambda}'$ be the corresponding lattice paths, and let $\{s_1, s_2, \ldots, s_r\}$ be the set of lattice points common to λ and λ'.

We assume that the s_i are labelled so that $s_1 = (0,0)$, $s_r = (n,m)$ and as we move along $\bar{\lambda}$ from s_1 to s_r, we reach s_i before s_{i+1}. Consider the pair of path segments $\bar{\lambda}(s_i, s_{i+1})$ (defined to be the portion of $\bar{\lambda}$ between (and including) s_i and s_{i+1}) and $\bar{\lambda}'(s_i, s_{i+1})$. We will call the closed region bounded by these two segments an olive, provided that the region is non-degenerate (i.e., $\bar{\lambda}(s_i, s_{i+1})$ and $\bar{\lambda}'(s_i, s_{i+1})$ do not coincide). Let $O_1, O_2, \ldots, O_t$ be the set of olives formed by $\bar{\lambda}$ and $\bar{\lambda}'$. The upper path segment bounding O_k we denote by O_k^+; the lower we denote by O_k^-. Note that, given $\bar{\lambda} \cup \bar{\lambda}'$, the path $\bar{\lambda}$ can be determined by specifying which O_i contribute O_i^+ to $\bar{\lambda}$ and consequently, which O_j contribute O_j^- to $\bar{\lambda}$.

We want to show that for each $\bar{\lambda} \in \hat{Z}' \cap \hat{W}$ with $\bar{\lambda}' \in \hat{Z} \cap \hat{W}'$, we can associate a unique $\bar{\mu} \in \hat{Z} \cap \hat{Z}'$ with $\bar{\mu}' \in \hat{W} \cap \hat{W}'$. In fact, $\bar{\mu}$ and $\bar{\mu}'$ will be constructed from the path segments of $\bar{\lambda}$ and λ' so that $\bar{\mu} \cup \bar{\mu}' = \bar{\lambda} \cup \bar{\lambda}'$. The rule for obtaining $\bar{\mu}$ (and consequently $\bar{\mu}'$) is as follows:

> Let $\bar{\mu}$ be the same as $\bar{\lambda}$ except that whenever an olive O_k is intersected by a barrier of Z or W, we let $O_k^+ \in \bar{\mu}$.

In the example illustrated in Figure 4, O_2 is penetrated (from below) by an x-barrier in Z-W, and O_4 is penetrated (from the left) by a y-barrier in W-Z. Note that λ always contains the lower boundaries O_k of the penetrated olives O_k. To obtain μ, we substitute O_2^+, O_4^+ for O_2^-, O_4^- in the path $\bar{\lambda}$

To show that $\bar{\mu} \in \hat{Z} \cap \hat{Z}'$ and that the complementary path $\bar{\mu}' \in \hat{W} \cap \hat{W}'$ we need only verify that $\bar{\mu}$ and $\bar{\mu}'$ clear their respective sets of barriers in $Z \cup Z'$ and $W \cup W'$ respectively in that section.

Suppose O_k is penetrated (from below) by an x-barrier in Z-W, such as the O_2 in Figure 4. Then λ contains O_k^- and $\bar{\lambda}'$ contains O_k^+. We want to argue that O_k^+ must clear Z and Z', while O_k^- must clear W and W'. First of all, if O_k^+ clears W' then it clears W'_Y and hence Z'_Y. Secondly, O_k^+ clears Z'_X since O_k^- clears Z'. It follows that O_k^+ clears both Z and Z' as desired. The fact that O_k clears W and W' can be shown in the same way.

Similarly, if O_k is penetrated by a y-barrier in W-Z, such as the O_4 in Figure 4, then assigning O_k^+ to $\bar{\mu}$ and O_k^- to $\bar{\mu}'$ will enable $\bar{\mu}$, $\bar{\mu}'$ to clear their respective barriers.

The mapping $(\bar{\lambda},\bar{\lambda}') \to (\bar{\mu},\bar{\mu}')$ is 1-1, since the path $\bar{\lambda}$ can be reconstructed from $\bar{\mu}$ by substituting O_k^- for O_k^+ in those olives O_k penetrated by a barrier of Z or W. This completes the proof of Theorem 1. □

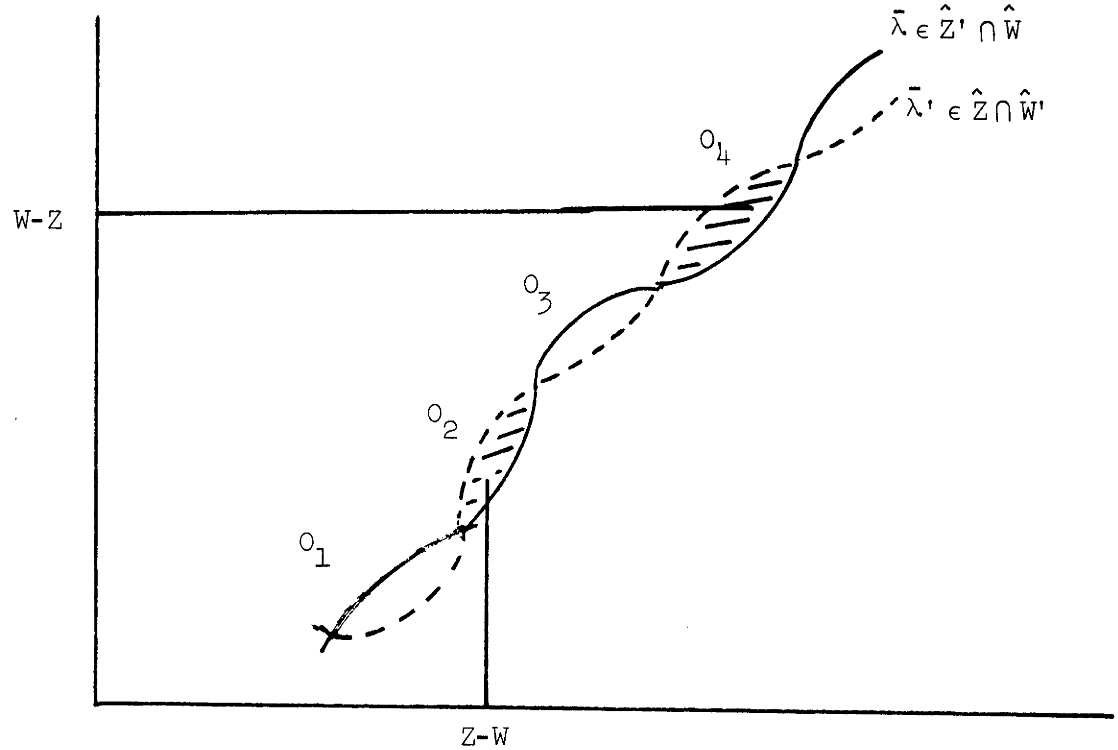

Figure 4. Olives which are penetrated by an x-barrier in Z-W and a y-barrier in W-Z.

3. Extension to Disjunctions of Partial Orders.

In this section we will consider pairs of cross relations (Z, W) on $A = \{a_1 < a_2 < \ldots < a_m\}$ and $B = \{b_1 < b_2 < \ldots < b_n\}$, when Z consists of just **x**-barriers and W consists of just **y**-barriers. However, we now incorporate the concept of a disjunction of a set of cross-relations. For a disjunction $z = \bigcup_i Z_i$ where $Z_i \subseteq (A \times B) \cup (B \times A)$, we let $\hat{z}$ denote $\bigcup_i \hat{Z}_i$. Suppose $\chi = \bigcup_i X_i$ and $\psi = \bigcup_j Y_j$ where $X_i \subseteq A \times B$ and $Y_j \subseteq B \times A$, with $\chi' = \bigcup_i X_i'$ and $\psi' = \bigcup_j Y_j'$ defined similarly. The analogue of Theorem 1 is the following:

Theorem 2. $|\hat{\chi} \cap \hat{\chi}'| \; |\hat{\psi} \cap \hat{\psi}'| \geq |\hat{\chi} \cap \hat{\psi}| \; |\hat{\chi}' \cap \hat{\psi}'|$.*/

As in the case of Theorem 1, here we can also derive as corollaries that $\dfrac{\Pr(\hat{\chi} \mid \hat{\chi}')}{\Pr(\hat{\psi} \mid \hat{\chi}')} \geq \dfrac{\Pr(\hat{\chi} \mid \hat{\psi}')}{\Pr(\hat{\psi} \mid \hat{\psi}')}$, that is, the ratio $\Pr(\hat{\chi} / \Pr \hat{\psi})$ is larger when conditioned on $\hat{\chi}'$ than when conditioned on $\hat{\psi}'$. For the special case that $\psi = \psi' = \emptyset$, we obtain

$$\Pr(\hat{\chi} \mid \hat{\chi}') \geq \Pr(\hat{\chi}) . \tag{1}$$

Proof of Theorem 2. As in the proof of Theorem 1, we will show that for each $\lambda \in \hat{\chi} \cap \hat{\psi}$ with $\lambda' \in \hat{\chi}' \cap \hat{\psi}'$, we can associate a unique $\bar{\mu} \in \hat{\chi} \cap \hat{\chi}'$ with $\bar{\mu} \in \hat{\psi} \cap \hat{\psi}'$. Furthermore, $\bar{\mu}$ and $\bar{\mu}'$ will be constructed from $\bar{\lambda}$ and $\bar{\lambda}'$ by interchanging certain path segments. We may assume without loss of generality that no X_i, X_i', Y_j, or Y_j' have a barrier which penetrates both $\bar{\lambda}$ and $\bar{\lambda}'$.

*/ We could of course write this as $|\hat{\chi} \cap \hat{\chi}'| \, |\hat{\psi} \cap \hat{\psi}'| \geq |\hat{\chi}' \cap \hat{\psi}| \, |\hat{\chi} \cap \hat{\psi}'|$ to make it resemble Theorem 1 more.

Let $O_1, O_2, \ldots, O_t$ be the set of olives formed by $\bar{\lambda}$ and $\bar{\lambda}'$. Thus $\bar{\lambda}$ corresponds to a subset $P \subset \{1,2,\ldots,t\} = T$ such that $\bar{\lambda}$ contains O_k^+ iff $k \in P$, and with this association $\bar{\lambda}'$ corresponds to the subset $Q = T-P = P^c$. For a given olive O_k, there may be various barriers which intersect it. For each X_i, let G_i denote the set $\{k \in T\text{: a barrier from } X_i \text{ intersects } O_k\}$. Similarly, define G_i' for X_i', H_i for Y_i and H_i' for Y_i'. Observe that

$$\begin{aligned} \bar{\lambda} \in \hat{\chi} \quad & \text{iff} \quad \bar{\lambda} \in \hat{X}_i \quad \text{for some } i \\ & \text{iff } P \supseteq G_i \quad \text{for some } i \\ & \text{iff } P \in [\mathscr{G}]_U \equiv \text{upper ideal in } 2^T \text{ generated by} \\ & \qquad \mathscr{G} = \{G_1, G_2, \ldots\} \quad . \end{aligned}$$

where the meaning of the last statement is as follows.

<u>Definition.</u> For a finite set T, let 2^T denote the collection of all subsets of T partially ordered by set inclusion (i.e., $C < D$ iff $C \supseteq D$). An <u>upper ideal</u> in 2^T is a subset $\mathcal{U} \subset 2^T$ such that if $S \in \mathcal{U}$ then any element S' higher in the partial order (i.e., $S \subseteq S'$) must also be in $\mathcal{U}$. Similarly, a <u>lower ideal</u> $\mathcal{L} \subseteq 2^T$ has the property that if $S \in \mathcal{L}$ and $S' \subseteq S$, then $S' \in \mathcal{L}$.

As above, we have

$$\begin{aligned} \bar{\lambda} \in \hat{\mathcal{Y}} \quad & \text{iff} \quad \bar{\lambda} \in \hat{Y}_j \quad \text{for some } j \\ & \text{iff} \quad P \subseteq H_j^c \quad \text{for some } j \\ & \text{iff } P \in [\mathscr{H}^c]_L \equiv \text{lower ideal in } 2^T \text{ generated by} \\ & \qquad \mathscr{H}^c = \{H_1^c, H_2^c, \ldots\} \quad . \end{aligned}$$

Now, what we are trying to show is that for each $\bar{\lambda} \in \hat{\chi} \cap \hat{\mathcal{Y}}$ with $\bar{\lambda}' \in \hat{\chi}' \cap \hat{\mathcal{Y}}'$ we can associate a unique $\bar{\mu} \in \hat{\chi} \cap \hat{\chi}'$ with $\bar{\mu}' \in \hat{\mathcal{Y}} \cap \hat{\mathcal{Y}}'$. Translating this into the language of ideals, we want:

For each $P \in [\mathcal{Y}]_U \cap [\mathcal{X}^c]_L$ with $P^c \in [\mathcal{Y}']_U \cap [\mathcal{X}'^c]_L$ there can be associated a unique $Q \in [\mathcal{Y}]_U \cap [\mathcal{Y}']_U$ with $Q^c \in [\mathcal{X}^c]_L \cap [\mathcal{X}'^c]_L$.

We claim that, in fact, we will be able to find such a mapping for <u>arbitrary</u> upper ideals $\mathcal{U}$, $\mathcal{U}'$ and lower ideals $\mathcal{L}$, $\mathcal{L}'$ in 2^T . In other words, there is a 1-1 mapping $(P,P^c) \to (Q,Q^c)$ such that if $P \in \mathcal{U} \cap \mathcal{L}$ and $P^c \in \mathcal{U}' \cap \mathcal{L}'$ then $Q \in \mathcal{U} \cap \mathcal{U}'$ and $Q^c \in \mathcal{L} \cap \mathcal{L}'$. Further, we will restrict the mapping so that

$$P \subseteq Q \quad . \tag{2}$$

If (2) holds then

$P \in \mathcal{U} \Rightarrow Q \in \mathcal{U}$ since $\mathcal{U}$ is an upper ideal,

$P^c \in \mathcal{L}' \Rightarrow Q^c \in \mathcal{L}'$ since $\mathcal{L}'$ is a lower ideal.

Thus, we want

$$\begin{matrix} P \in \mathcal{U} \cap \mathcal{L} \\ P^c \in \mathcal{U}' \cap \mathcal{L}' \end{matrix} \Rightarrow \begin{matrix} Q \in \mathcal{U}' \\ Q^c \in \mathcal{L} \end{matrix} \quad \text{with } P \subseteq Q \ .$$

We claim even further that we can find the required mapping for the more general domain

$$\begin{matrix} P \in \mathcal{L} \\ P^c \in \mathcal{U}' \end{matrix} \Rightarrow \begin{matrix} Q \in \mathcal{U}' \\ Q^c \in \mathcal{L} \end{matrix} \quad \text{with } P \subseteq Q \ .$$

But notice that if $\mathcal{U}'$ is an upper ideal then $\mathcal{U}'^c$ is a lower ideal. Thus, the condition

$$\begin{matrix} P \in \mathcal{L} \\ P^c \in \mathcal{U}' \end{matrix} \Rightarrow \begin{matrix} Q \in \mathcal{U}' \\ Q^c \in \mathcal{L} \end{matrix} \quad \text{with } P \subseteq Q$$

becomes

$$P \in \mathcal{L} \cap \mathcal{U}'^c \equiv \mathfrak{w} \Rightarrow Q^c \in \mathfrak{w} \quad \text{with } P \subseteq Q$$

where $\mathfrak{w}$, being the intersection of two lower ideals, is also a lower ideal. Of course,

$$P \subseteq Q \quad \text{iff} \quad P \cap Q^c = \emptyset \quad .$$

Thus, the theorem will be proved if we show the following result, which is actually of independent interest:

> For an arbitrary lower ideal $\mathfrak{w}$ in 2^T, there is always a permutation $\pi: \mathfrak{w} \to \mathfrak{w}$ such that for all $w \in \mathfrak{w}$, $w \cap \pi(w) = \emptyset$.

For each $x \in \mathfrak{w}$, let $d(x)$ denote the set $\{w \in \mathfrak{w}: x \cap w = \emptyset\}$. By Hall's Theorem [2], it is enough to show that

$$\left| \bigcup_{x \in \mathcal{A}} d(x) \right| \geq |\mathcal{A}|$$

for all $\mathcal{A} \subseteq \mathfrak{w}$. In fact, for $\mathcal{A} \subseteq \mathfrak{w}$, let $d_{\mathcal{A}}(x)$ denote $d(x) \cap [\mathcal{A}]_L$. What we will actually show is the stronger assertion

$$\left| \bigcup_{x \in \mathcal{A}} d_{\mathcal{A}}(x) \right| \geq |\mathcal{A}| \qquad (2)$$

for any $\mathcal{A} \subset 2^T$. So, suppose $\mathcal{A} = \{S_1, \ldots, S_k\}$ with $S_i \subseteq T$. Thus,

$$\begin{aligned} y \in \bigcup_{x \in \mathcal{A}} d_{\mathcal{A}}(x) \quad & \text{iff} \quad y \in [\mathcal{A}]_L \text{ and } y \cap x = \emptyset \text{ for some } x \in \mathcal{A}, \\ & \text{iff} \quad y \subseteq S_i \text{ for some } i \text{ and } y \cap S_j = \emptyset \text{ for some } j, \\ & \text{iff} \quad y \subseteq S_i - S_j \text{ for some } i, j. \end{aligned}$$

Therefore, if we can in fact show that there are always at least k different sets of the form $S_i - S_j$ then (6) will follow. However, this is exactly the result of Marica and Schönheim [4]. Hence (3) holds and the theorem follows. □

Theorem 2 can be generalized slightly by allowing the partial order $(P, <)$ underlying $\hat{\chi}$, $\hat{y}$, $\hat{\chi}'$, $\hat{y}'$ to be more than just $A \cup B$, i.e., P may itself include relations of the form $a_i < b_j$ and $b_k < a_\ell$. In this case, all such relations can also be interpreted as barriers which cannot be crossed by a linear extension $\bar{\sigma}$ of P. Since both paths $\bar{\lambda}$ and $\bar{\lambda}'$ avoid all these barriers then so will any path $\bar{\mu}$, $\bar{\mu}'$ constructed from their path segments.

We should point out that if we weaken the hypotheses on the structure of $(P, <)$ even slightly then formula (2) (and even (1)) can fail. To see this, consider the following partial order $(P, <)$ on the set $\{a_1, a_2, b_1, b_2, c\}$ as shown in Figure 5.

Choose $X = X_1 = \{(1,1)\}$, $X' = X_1' = \{(2,2)\}$, and all other X_i, X_i', Y_j, Y_j' to be $\emptyset$. An easy enumeration yields

$$|\Lambda| = 8 \ , \quad |\hat{X}| = 3 = |\hat{X}'| \ , \quad |\hat{X} \cap \hat{X}'| = 1 \ .$$

Thus,

$$\Pr(\hat{X} \mid \hat{X}') = \frac{1}{3} < \frac{3}{8} = \Pr(\hat{X})$$

which violates (1). Therefore, the assumption that P can be covered by two linear orders seems to be essential for the general validity of formula (2).

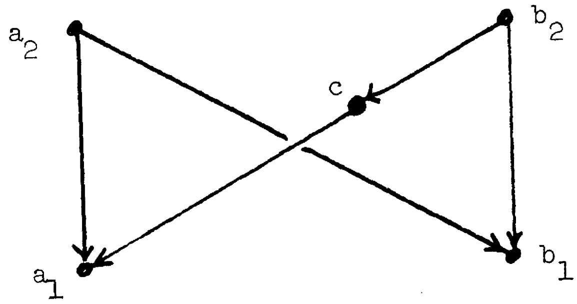

Figure 5. An example violating formulas (1) and (2).

4. Conclusions.

We end with some remarks on the Conjecture $Pr(E \mid P') > Pr(E \mid P)$ which is left open in this paper. By Corollary 2 to Theorem 1, we know that $Pr(E \mid P' \cup S) \geq Pr(E \mid P \cup S)$ for any $S = \{a_{i_1} < a_{i_2} < \ldots < a_{i_m}; b_{j_1} < b_{j_2} < \ldots < b_{j_n}\}$. It is tempting to try to prove the Conjecture by making use of the facts

$$Pr(E \mid P) = \sum_S Pr(S \mid P) . Pr(E \mid P \cup S) \text{ and } Pr(E \mid P') = \sum_S Pr(S \mid P') \cdot Pr(E \mid P' \cup S) .$$

However, as warned by Simpson's paradox [6], such a direct inference is not possible, and the validity of the Conjecture must depend on deeper properties of partial orders. A different type of monotonicity property for distributive lattices, usually called the FKG inequalities, has been treated in the literature [1],[5]. These may well be relevant to the eventual resolution of our problem.

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